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SOLUTION OF FRACTIONAL HEAT-LIKE AND FRACTIONAL WAVE-LIKE EQUATION BY USING MODERN STRATEGY

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Abstract: This paper introduces a novel form of the Adomian decomposition (ADM) method for solving fractional-order heat-like and wave-like equations with starting and boundary value problems. The derivations are provided in the sense of Caputo. In order to help understanding, the generalised formulation of the current approach is provided. Several numerical examples of fractional-order diffusion-wave equations (FDWEs) are solved using the suggested method in this context. In addition to examining the applicability of the suggested method to the solving of fractional-order heat-like and wave-like equations, a graphical depiction of the solutions to three instructive cases was constructed. Solution graphs were arrived at for integer and fractional-order problems. The derived and exact solutions to integer-order problems were found to be in excellent agreement. The subject of the present research endeavour is the convergence of fractional-order solutions. This strategy is considered to be the most successful way of addressing fractional-order initial-boundary value issues in science and engineering. This strategy is presented here.

Key words: fractional-order heat-like and wave-like equations, initial-boundary value problems, Adomian decomposition method

1. INTRODUCTION

Fractional calculus is the study of derivatives and integrals of fractional order. In addition to fluid dynamics, viscoelasticity, chemistry, physics and finance, these approaches may also be used in other fields. Fractional differential equations are used in several scientific and engineering fields.

The fractional method is currently regarded as the most powerful modelling tool in wave propagation, anomalous diffusion tools, turbulence and mechanics [1, 2, 3]. It is an extension of the conventional integer-order partial differential equations and fractional partial differential equations (FPDEs). In the past 10 years, scientists and engineers have paid a lot of attention to nonlinear equations due to the fact that nonlinearity is present in almost all physical situations. Nonlinear partial differential equations of fractional order are used in chemistry, biology, physics, vibration, acoustics, signal processing, electromagnetics, polymeric materials and fluid dynamics, as well as superconductivity, optics and quantum mechanics [4–7]. For the description of many elements of natural phenomena, such as the dynamics of complex materials, quasi-chaotic dynamical systems and random walks with memory, FPDEs are better suited than traditional PDEs [8,9,10].

Since FDEs typically lack accurate analytical solutions, the manner of solving these equations in approximate and numerical ways has been subject to much investigation [11–13]. Methods such as variational iteration, Adomian decomposition (ADM), homotopy perturbation, Lagrange multiplier technique and others are used to provide analytical approximations for linear and nonlinear FDEs. It is necessary to build an effective and user-friendly method for solving these equations. The ADM method may be used to solve this issue with ordinary, partial and nonlinear differential equations [14–17].

A broad class of linear or nonlinear differential equations has been approximated using the decomposition approach [18,19]. The method's use for fractional differentia equations has recently been broadened [20–23]. Researchers are expected to work on the resolution of fractional-order diffusion-wave equations (FDWEs) using the ADM approach, which is a novel technology [24]. FDWEs are the most important type of anomalous diffusion equation derived from classical diffusion-wave equations [25]. Anh and Leonenko [26] provide the mean-square solution comprising the Green function and the spectral representation of FDWEs. Ali [27] uses a new ADM method to solve FDWEs having both starting and ending conditions.

In this paper, we will concentrate on solving FDWEs using a novel ADM methodological approach. Initial and boundary value issues are dealt with using ADM and its variants. The solutions to a few test problems and their graphical representation are arrived at using the Mathematica program to demonstrate the applicability of the current technique.

The remainder of the article is structured as follows: In Section 2, the details of the new iterative method, theorem proofs and its convergence are discussed. The model's description and how it is used to obtain the exact analytical solutions to the specified fractional heat-like and wave-like equations are also discussed. In Section 3, we demonstrate the proposed method's reliability, convergence and efficiency using four exemplary instances. In Section 4, a debate over the examples provided in the article is presented, with the aid of some graphs and tables, and related discussion. Finally, Section 5 describes the conclusions we have drawn from the study.

Definition 1: provides the Reimann–Liouville (RL) integral operator of arbitrary order $\tau(\tau \geq 0)$ for a function $\chi(\beta)$ [28, 29]

$$J^{\tau}\chi(\beta) = \frac{1}{\Gamma(\tau)}\int_0^1 (\beta - \iota)^{\tau-1}\chi(\iota)d\iota. \tag{1}$$

The gamma function that permits Eq. (10) to converge on $(0, \infty)$ point-wise is $\Gamma(\tau) = \int_0^{\infty} \beta^{\tau-1}e^{-\beta}d\beta$. The integral operator (RL) has the following properties.

$$\begin{cases} J^{\tau}J^{\mu}\chi(\beta) = J^{\mu}J^{\tau}\chi(\beta), \\ J^{\tau}J^{\mu}\chi(\beta) = J^{\mu+\tau}\chi(\beta), \\ \quad J^{\tau}\beta^{\mu} = \frac{\Gamma(\mu+)}{\Gamma(\mu+\tau+1)}. \end{cases}$$

Definition 2: For the function $\chi(\beta)$, the Caputo operator of fractional order an is shown [28].

$$\mathrm{D}^{\tau}\chi(\beta) = \frac{1}{\Gamma(\varepsilon-\tau)}\int_0^t \frac{\chi^{(\varepsilon)}(\iota)}{(\beta-\iota)^{\tau+1-\varepsilon}}d\iota \quad \varepsilon - 1 < \tau < \varepsilon, \varepsilon = [\varepsilon] + 1. \tag{2}$$

The following conditions must be met by Eq. (2).

$$D^{\tau}\chi^{k} = \begin{cases} 0. \, k \in N, k < |\tau|, \\ \frac{\Gamma(k+1)}{1+k-\tau}k^{k-\tau}. \end{cases}$$

Definition 3: The Mittage–Leffler function [30] is expressed as:

$$M_{\tau}(\beta) = \sum_{n=0}^{\infty} \frac{\beta^n}{\Gamma(n\tau+1)}, \tau > 0, \beta \in C. \tag{3}$$

2. BASED CONCEPT ON ADM

Adomian devised this approach for solving differentials and integrating differential problems in 1994. The following process can be used to illustrate the current method. Let

$$\phi\big(\chi(\gamma)\big) = \psi(\gamma) \tag{4}$$

If $\psi(x)$ stands for the known function, ϕ stands for the differential operator, which may be broken down as follows:

$$\phi(\chi) = L_{\chi} + R_{\chi} + N_{\chi}\,, \tag{5}$$

where R and N are linear and nonlinear terms, respectively, and L is the invertible operator of the largest derivative. Eq. (4) therefore has the following representation:

$$L_{\chi} + R_{\chi} + N_{\chi} = \psi \tag{6}$$

Taking L^{-1} of Eq. (5), we have

$$\chi = \eta + L^{-1}(\psi) - L^{-1}(R\chi) - L^{-1} \tag{7}$$

The integration constant η is used here. The following are infinite series representations of the ADM

solution:

$$\chi = \sum_{\rho=0}^{\infty} \chi_{\rho} \tag{8}$$

The nonlinear term N_{χ} is denoted by A_{ρ} Adomian polynomials and is defined as:

$$N_{\chi} = \sum_{\rho=0}^{\infty} A_{\rho}, \tag{9}$$

We can compute A_{ρ} with the aid of the formula below.

$$A_{\rho} = \frac{1}{\rho!}\frac{d^{\rho}}{d\lambda^{\rho}}N(\sum(\lambda^{\rho}\chi_K)), \qquad \rho = 0,1,2, \ldots$$

The following connection is used to represent the solution of Eq. (4):

$$\begin{cases} \chi_0 = \eta + L^{-1}(\psi)\,, \rho = 0 \\ \chi_{\rho+1} = L^{-1}\big(R\chi_{\rho}\big) - L^{-1}\big(A_{\rho}\big), \rho \geq 0 \end{cases} \tag{10}$$

3. MODIFIED ADM FOR INTINAL-BOUNDARY VALUE PROBLEMS

Consider the one-dimensional differential equation below to convey the key concept of treating initial and boundary conditions with the ADM approach for resolving initial-boundary value difficulties.

$$D_{\beta}^{\mathrm{T}}\chi(\gamma, \beta) = D_{\gamma\gamma}\chi(\gamma, \beta) + R(\gamma, \beta), 0 < \gamma < 1,$$
$$\beta > 0, 1 < \mathrm{T} < 2\,. \tag{11}$$

The IC for Eq. (11) is of the following type:

$$\chi(\gamma, 0) = \delta_0(\gamma), \chi_{\beta}(\gamma, 0) = \delta_1(\gamma), 0 \leq \gamma \leq 1$$

Hence, the BC is defined as follows:

$$\chi(0, \beta) = \lambda_0(\beta), \chi\,(1, \beta) = \lambda_1(\beta), \qquad \beta \geq 0$$

The operator form of ADM Eq. (11) is:

$$L_{\chi} = D_{\gamma\gamma}\chi(\gamma, \beta) + R(\gamma, \beta) \tag{12}$$

where L is defined as,

$$L = \frac{\partial^{\mathrm{T}}}{\partial\beta^{\mathrm{T}}}\,, 1 < \mathrm{T} \leq 2.$$

Hence, L^{-1} is defined as,

$$L^{-1}(\,.\,) = I^{\mathrm{T}}(\,.\,)d\beta. \tag{13}$$

Applying L^{-1} to Eq. (12), we obtain:

$$\chi(\gamma, \beta) = \eta + L^{-1}\left(D_{\gamma\gamma}\chi(\gamma, \beta) + R(\gamma, \beta)\right) \tag{14}$$

when using ADM, the initial approximation becomes more accurate.

$$\chi_0(\gamma, \beta) = \chi(\gamma, 0) + \beta\left(\partial_{\beta}\chi(\gamma, 0)\right) + L^{-1}\big(R(\gamma, \beta)\big).$$

The iteration formula becomes more powerful when used with the new ADM technique.

$$\chi_{\rho+1}(\gamma, \beta) = L^{-1}\big(D_{\gamma\gamma}\chi_{\rho}^{*}\big), \rho = 0,1,2, \ldots \tag{15}$$

where the new χ_{ρ}^{*} is then computed using the newly suggested method.

$$\chi_{\rho}^{*} = \chi_{\rho}(\gamma, \beta) + (1-\gamma)\left(\lambda_0(\beta) - \chi_{\rho}(0, \beta)\right) +$$
$$\gamma\left(\lambda_1(\beta) - \chi_{\rho}(1, \beta)\right), \rho = 0,1,2, \ldots \tag{16}$$

It is clear that the new successive initial solutions χ_{ρ}^{*} of Eq. (11) satisfy both the initial and boundary conditions when

$\rho = 0,1,2,\ldots$

At $\beta = 0\,,\quad \chi_{\rho}^{*}(\gamma, 0) = \chi_{\rho}(\gamma, 0),$

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$$\gamma = 0, \quad \chi_\rho^*(0,\beta) = \lambda_0(\beta),$$
$$\gamma = 1, \quad \chi_\rho^*(1,\beta) = \lambda_1(\beta).$$

Theorem 1. (Uniqueness theorem) We consider the following general time FPDE.

$$L^{\mathrm{T}}\chi(\gamma,\beta) + R\chi(\gamma,\beta) + N\chi(\gamma,\beta) = \psi(\gamma,\beta), \tag{17}$$

$m-1 < \mathrm{T} \le m, \gamma > 0, \beta > 0$ where L is a fractional-order derivative, R is a linear differential operator, is a nonlinear operator and ψ is a source term, and where $R(\chi)$ and $N(\chi)$ satisfy the Lipschitz condition with the constants L_1 and L_2 Then, Eq. (17) has a unique solution whenever $0 < K < 1$ for $K = \frac{(L_1+L_2)\beta^{\mathrm{T}}}{\Gamma(\mathrm{T}+1)}$.

Proof: Let θ be the Banach space of all continuous functions on $I = [0,T]$ with the norm $\| \chi(\beta) \| = max|\chi(\beta)|$. We define a mapping $\omega : \theta \to \theta$, where

$$\omega(\chi(\beta)) = \varphi(\beta) + \mathrm{T}_\psi(\gamma,\beta) - J[R\psi(\gamma,\beta)] - J^{\mathrm{T}}N\psi(\gamma,\beta)\} \tag{18}$$

Let $\chi, \bar{\chi} \in \theta$

$$\begin{aligned}\|\omega_\chi - \omega_{\bar{\chi}}\| &= max|\omega_\chi - \omega_{\bar{\chi}}|,\\ &= |\varphi(\beta) + J^{\mathrm{T}}\psi(\gamma,\beta) - J^{\mathrm{T}}[R\psi(\gamma,\beta)] - [J^{\mathrm{T}}N\psi(\gamma,\beta)] \\ &\quad - \varphi(\beta) - J^{\mathrm{T}}\psi(\gamma,\beta) + J^{\mathrm{T}}[R\psi(\gamma,\beta)] \\ &\quad + [J^{\mathrm{T}}N\psi(\gamma,\beta)]| \\ &= |J^{\mathrm{T}}[R\psi(\gamma,\beta) - R\psi(\gamma,\beta)]| + J^{\mathrm{T}}[N\psi(\gamma,\beta) - N\psi(\gamma,\beta)],\end{aligned} \tag{19}$$

Now, we suppose that $\mathrm{R}(\chi)$ and $\mathrm{N}(\chi)$ satisfy the Lipschitz condition with the constants L_1 and L_2.

Therefore,

$$\|\omega_\chi - \omega_{\bar{\chi}}\| \le max[J^{\mathrm{T}}|R\psi(\gamma,\beta) - R\psi(\gamma,\beta)| + J^{\mathrm{T}}|N\psi(\gamma,\beta) - N\psi(\gamma,\beta)|],$$

$$\|\omega_\chi - \omega_{\bar{\chi}}\| \le max[L_1 J^{\mathrm{T}}|\psi(\gamma,\beta) - \psi(\gamma,\beta)| + L_2 J^{\mathrm{T}}|\psi(\gamma,\beta) - \psi(\gamma,\beta)|],$$

$$\|\omega_\chi - \omega_{\bar{\chi}}\| \le (L_1 + L_2)\|\psi(\gamma,\beta) - \psi(\gamma,\beta)\| \frac{\beta^{\mathrm{T}}}{\Gamma(\mathrm{T}+1)}$$
$$\le K\|\psi(\gamma,\beta) - \psi(\gamma,\beta)\|, \text{ where } K = \frac{(L_1+L_2)\beta^{\mathrm{T}}}{\Gamma(\mathrm{T}+1)}$$

4. APPLICATIONS

In this part, we will show how to solve various exemplary problems utilising the new ADM-based method.

Example 1. We examine a fractional heat-like equation in one dimension. [31]

$$D_\beta^\tau \chi = \frac{1}{2}\gamma^2 \frac{\partial^2 \chi}{\partial \gamma^2}, \quad 0 < \tau, \gamma \le 1, \beta > 0. \tag{21}$$

Subject to the BC:

$$\chi(0,\beta) = 0, \quad \chi(1,\beta) = e^\beta, \ \beta > 0. \tag{22}$$

and IC:

$$\chi(\gamma,0) = \gamma^2. \tag{23}$$

By using ADM, Eq. (21) can be written in the form,

$$L_\chi = \frac{1}{2}\gamma^2 \frac{\partial^2 \chi}{\partial \gamma^2}. \tag{24}$$

where $\mathrm{L} = \mathrm{D}_\beta^\tau, \ 0 < \tau \le 1.$

Taking L^{-1}, of Eq. (24), we find that:

$$\chi(\gamma,\beta) = \chi(\gamma,0) + L^{-1}\left[\frac{1}{2}\gamma^2\chi_{\gamma\gamma}\right].$$

Using the initial approximation, we find that:

$$\chi_0(\gamma,\beta) = \gamma^2. \tag{25}$$

By use of the new technique of initial approximation χ_ρ^*, we have

$$\chi_{\rho+1} = \frac{1}{2}\gamma^2 L^{-1}[\chi_{\rho\gamma\gamma}^*], \quad \rho = 0,1,2,3,\dots \tag{26}$$

By applying a new approximation χ_ρ^*, we have:

$$\chi_\rho^*(\gamma,\beta) = \chi_\rho(\gamma,\beta) + (1-\gamma)[0 - \chi_\rho(0,\beta)] + \gamma[e^\beta - \chi_\rho(1,\beta)], \quad \rho = 0,1,2,\dots \tag{27}$$

Let $\rho = 0$; we then obtain:

$$\chi_0^*(\gamma,\beta) = \chi_0(\gamma,\beta) + (1-\gamma)[0 - \chi_0(0,\beta)] + \gamma[e^\beta - \chi_0(1,\beta)].$$

Using Eq. (26), we can obtain:

$$\chi_1(\gamma,\beta) = \frac{\gamma^2\beta^\tau}{\Gamma(\tau+1)},$$

Let $\rho = 1$; we then obtain:

$$\chi_1^*(\gamma,\beta) = \chi_1(\gamma,\beta) + (1-\gamma)[0 - \chi_1(0,\beta)] + \gamma[e^\beta - \chi_1(1,\beta)].$$

Using Eq. (26), we can obtain:

$$\chi_2(\gamma,\beta) = \frac{\gamma^2\beta^{2\tau}}{\Gamma(2\tau+1)},$$

Let $\rho = 2$; we then obtain:

$$\chi_2^*(\gamma,\beta) = \chi_2(\gamma,\beta) + (1-\gamma)[0 - \chi_2(0,\beta)] + \gamma[e^\beta - \chi_2(1,\beta)].$$

Using Eq. (26), we can obtain:

$$\chi_3(\gamma,\beta) = \frac{\gamma^2\beta^{3\tau}}{\Gamma(3\tau+1)},$$

Thus, the new ADM solution for Eq. (21) can be written in a series form:

$$\chi(\gamma,\beta) = \chi_0(\gamma,\beta) + \chi_1(\gamma,\beta) + \chi_2(\gamma,\beta) + \chi_3(\gamma,\beta) + \cdots = \gamma^2 M_\tau(\beta). \tag{28}$$

where $\mathrm{M}_\tau(\beta)$ is the Mittag-Leffler function.

If $\tau = 1$, we obtain:

$$\chi(\gamma,\beta) = \gamma^2\left(1 + \beta + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \cdots\right) = \gamma^2 e^\beta. \tag{29}$$

which is the exact solution of Eq. (21).

Example 2. We examine a fractional wave-like equation in one dimension [32].

$$D_{\beta\beta}^\tau \chi = \frac{1}{2}\gamma^2 \frac{\partial^2 \chi}{\partial \gamma^2}, \quad 0 < \gamma < 1, 0 < \tau \le 2, \beta > 0. \tag{30}$$

Subject to the BC:

$$\chi(0,\beta) = 0, \quad \chi(1,\beta) = 1 + \sinh\beta, \ \beta > 0. \tag{31}$$

and IC:

$$\chi(\gamma,0)=\gamma,\ \chi_{\beta}(\gamma,0)=\gamma^{2} \tag{32}$$

By using ADM, Eq. (30) can be written in the form,

$$L_{\beta\chi}=\frac{1}{2}\ \gamma^{2}\frac{\partial^{2}\chi}{\partial\gamma^{2}}. \tag{33}$$

where $\mathrm{L}=\mathrm{D}_{\beta}^{\tau},\ 1<\tau\leq 2$.

Taking L_{β}^{-1} of Eq. (33) enables us to find that:

$$\chi(\gamma,\beta)=\chi(\gamma,0)+\beta\chi_{\beta}(\gamma,0)+L^{-1}\left[\frac{1}{2}\ \gamma^{2}\chi_{\gamma\gamma}\right].$$

Using the initial approximation, we find that:

$$\chi_{0}(\gamma,\beta)=\gamma+\gamma^{2}\beta. \tag{34}$$

By use of the new technique of initial approximation χ_{ρ}^{*}, we have:

$$\chi_{\rho+1}=\frac{1}{2}\ \gamma^{2}L^{-1}\left[\chi_{\rho\gamma\gamma}^{*}\right],\ \ \rho=0,1,2,3,\ldots \tag{35}$$

By applying a new approximation χ_{ρ}^{*}, we have:

$$\chi_{\rho}^{*}(\gamma,\beta)=\chi_{\rho}(\gamma,\beta)+(1-\gamma)\left[0-\chi_{\rho}(0,\beta)\right]+\gamma\left[1+\sinh\beta-\chi_{\rho}(1,\beta)\right],\ \ \rho=0,1,2,\ldots \tag{36}$$

Let $\rho=0$; we then obtain:

$$\chi_{0}^{*}(\gamma,\beta)=\chi_{0}(\gamma,\beta)+(1-\gamma)[0-\chi_{0}(0,\beta)]+\gamma[1+\sinh\beta-\chi_{0}(1,\beta)].$$

Using Eq. (35), we can obtain the following:

$$\chi_{1}(\gamma,\beta)=\frac{\gamma^{2}\beta^{\tau+1}}{\Gamma(\tau+2)},$$

Let $\rho=1$; this enables us to obtain the following:

$$\chi_{1}^{*}(\gamma,\beta)=\chi_{1}(\gamma,\beta)+(1-\gamma)[0-\chi_{1}(0,\beta)]+\gamma[1+\sinh\beta-\chi_{1}(1,\beta)].$$

Using Eq. (35), we can obtain:

$$\chi_{2}(\gamma,\beta)=\frac{\gamma^{2}\beta^{2\tau+1}}{\Gamma(2\tau+2)},$$

Let $\rho=2$; we may then obtain:

$$\chi_{2}^{*}(\gamma,\beta)=\chi_{2}(\gamma,\beta)+(1-\gamma)[0-\chi_{2}(0,\beta)]+\gamma[1+\sinh\beta-\chi_{2}(1,\beta)].$$

Using Eq. (35), we can obtain:

$$\chi_{3}(\gamma,\beta)=\frac{\gamma^{2}\beta^{3\tau+1}}{\Gamma(3\tau+2)},$$

Thus, the new ADM solution for Eq. (30) can be written in a series form:

$$\chi(\gamma,\beta)=\chi_{0}(\gamma,\beta)+\chi_{1}(\gamma,\beta)+\chi_{2}(\gamma,\beta)+\chi_{3}(\gamma,\beta)+\cdots$$
$$=\gamma+\gamma^{2}\beta+\gamma^{2}\left[\frac{\beta^{\tau+1}}{\Gamma(\tau+2)}+\frac{\beta^{\tau+3}}{\Gamma(\tau+4)}+\frac{\beta^{\tau+5}}{\Gamma(\tau+6)}+\cdots\right], \tag{37}$$

If $\tau=2$, we obtain:

$$\chi(\gamma,\beta)=\gamma+\gamma^{2}\left(\beta+\frac{\beta^{3}}{3!}+\frac{\beta^{5}}{5!}+\frac{\beta^{7}}{7!}+\cdots\right)$$
$$=\gamma+\gamma^{2}\sinh\beta. \tag{38}$$

which is the exact solution of Eq. (30).

Example 3. Consider the following two-dimensional linear [32].

$$D_{\beta}^{\tau}\chi=\frac{\partial^{2}\chi}{\partial\gamma^{2}}+\frac{\partial^{2}\chi}{\partial\eta^{2}},\ \ 0<\gamma,\eta<2\pi\ ,0<\tau\leq 1,\beta>0. \tag{39}$$

Subject to the BC:

$$\begin{aligned}\chi(0,\eta,\beta)&=\chi(2\pi,\eta,\beta)=0\\ \chi(\gamma,0,\beta)&=\chi(\gamma,2\pi,\beta)=0\end{aligned} \tag{40}$$

and IC:

$$\chi(\gamma,\eta,\beta)=\sin\gamma\,\sin\eta \tag{41}$$

By using ADM, Eq. (39) can be written in the form,

$$L_{\chi}=\frac{\partial^{2}\chi}{\partial\gamma^{2}}+\frac{\partial^{2}\chi}{\partial\eta^{2}}, \tag{42}$$

where $\mathrm{L}=\mathrm{D}_{\beta}^{\tau},\ 0<\tau\leq 1$.

Taking L^{-1} of Eq. (42), we find that:

$$\chi(\gamma,\eta,\beta)=\chi(\gamma,\eta,0)+L^{-1}\left[\frac{\partial^{2}\chi}{\partial\gamma^{2}}+\frac{\partial^{2}\chi}{\partial\eta^{2}}\right]. \tag{43}$$

Using the initial approximation, we find that:

$$\chi_{0}(\gamma,\eta,\beta)=\sin\gamma\,\sin\eta. \tag{44}$$

By use of the new technique of initial approximation χ_{ρ}^{*}, we have:

$$\chi_{\rho+1}=L^{-1}\left[\chi_{\rho\gamma\gamma}^{*}+\chi_{\rho\gamma\gamma}^{*}\right],\ \ \rho=0,1,2,3,\ldots \tag{45}$$

By applying a new approximation χ_{ρ}^{*}, we have:

$$\chi_{\rho}^{*}(\gamma,\eta,\beta)=\chi_{\rho}(\gamma,\eta,\beta)+(1-\gamma)\left[\chi(0,\eta,\beta)-\chi_{\rho}(0,\eta,\beta)\right]+\gamma\left[\chi(2\pi,\eta,\beta)-\chi_{\rho}(2\pi,\eta,\beta)\right]+(1-\eta)\left[\chi(\gamma,0,\beta)-\chi_{\rho}(\gamma,0,\beta)\right]+\eta\left[\chi(\gamma,2\pi,\beta)-\chi_{\rho}(\gamma,2\pi,\beta)\right],\ \ \rho=0,1,2,\ldots \tag{46}$$

Let $\rho=0$; we then obtain:

$$\chi_{0}^{*}(\gamma,\eta,\beta)=\chi_{0}(\gamma,\eta,\beta)+(1-\gamma)[\chi(0,\eta,\beta)-\chi_{0}(0,\eta,\beta)]+\gamma[\chi(2\pi,\eta,\beta)-\chi_{0}(2\pi,\eta,\beta)]+(1-\eta)[\chi(\gamma,0,\beta)-\chi_{0}(\gamma,0,\beta)]+\eta[\chi(\gamma,2\pi,\beta)-\chi_{0}(\gamma,2\pi,\beta)]=\sin\gamma\,\sin\eta\,. \tag{47}$$

Using Eq. (45), we obtain:

$$\chi_{1}(\gamma,\eta,\beta)=L^{-1}\left[\chi_{0\gamma\gamma}^{*}+\chi_{0\gamma\gamma}^{*}\right]=-2\sin\gamma\,\sin\eta\,\frac{\beta^{\tau}}{\Gamma(\tau+1)}, \tag{48}$$

Let $\rho=1$; c

$$\chi_{1}^{*}(\gamma,\eta,\beta)=\chi_{1}(\gamma,\eta,\beta)+(1-\gamma)[\chi(0,\eta,\beta)-\chi_{1}(0,\eta,\beta)]+\gamma[\chi(2\pi,\eta,\beta)-\chi_{1}(2\pi,\eta,\beta)]+(1-\eta)[\chi(\gamma,0,\beta)-\chi_{1}(\gamma,0,\beta)]+\eta[\chi(\gamma,2\pi,\beta)-\chi_{1}(\gamma,2\pi,\beta)]=-2\sin\gamma\,\sin\eta\,\frac{\beta^{\tau}}{\Gamma(\tau+1)}. \tag{49}$$

Using Eq. (49), we obtain:

$$\chi_{2}(\gamma,\eta,\beta)=4\sin\gamma\,\sin\eta\,\frac{\beta^{2\tau}}{\Gamma(2\tau+1)}, \tag{50}$$

Let $\rho=2$; we thus obtain:

$$\chi_{2}^{*}(\gamma,\eta,\beta)=\chi_{2}(\gamma,\eta,\beta)+(1-\gamma)[\chi(0,\eta,\beta)-\chi_{2}(0,\eta,\beta)]+\gamma[\chi(2\pi,\eta,\beta)-\chi_{2}(2\pi,\eta,\beta)]+(1-\eta)[\chi(\gamma,0,\beta)-\chi_{2}(\gamma,0,\beta)]+\eta[\chi(\gamma,2\pi,\beta)-\chi_{2}(\gamma,2\pi,\beta)]=4\sin\gamma\,\sin\eta\,\frac{\beta^{2\tau}}{\Gamma(2\tau+1)}. \tag{51}$$

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Us Eq. (45), we obtain:

$$\chi_3(\gamma,\eta,\beta) = -8\sin\gamma\,\sin\eta\,\frac{\beta^{3\tau}}{\Gamma(3\tau+1)}\,, \tag{52}$$

Thus, the new ADM solution for Eq. (39) can be written in a series form:

$$\chi(\gamma,\eta,\beta) = \chi_0(\gamma,\eta,\beta) + \chi_1(\gamma,\eta,\beta) + \chi_2(\gamma,\eta,\beta) + \chi_3(\gamma,\eta,\beta) + \cdots = \sin\gamma\,\sin\eta - 2\sin\gamma\,\sin\eta\,\frac{\beta^{\tau}}{\Gamma(\tau+1)} + 4\sin\gamma\,\sin\eta\,\frac{\beta^{2\tau}}{\Gamma(2\tau+1)} - 8\sin\gamma\,\sin\eta\,\frac{\beta^{3\tau}}{\Gamma(3\tau+1)} + \cdots,$$

$$= \sin\gamma\,\sin\eta\left[1 - 2\frac{\beta^{\tau}}{\Gamma(\tau+1)} + 4\frac{\beta^{2\tau}}{\Gamma(2\tau+1)} - 8\frac{\beta^{3\tau}}{\Gamma(3\tau+1)} + \cdots\right] \tag{53}$$

If $\tau = 1$, we obtain:

$$\chi(\gamma,\eta,\beta) = \sin\gamma\,\sin\eta\; e^{-2\beta} \tag{54}$$

which is the exact solution of Eq. (39).

Example 4. We examine a fractional wave-like equation in two dimensions [33].

$$D^{\tau}_{\beta\beta}\chi = \frac{1}{12}\left[\gamma^2\frac{\partial^2\chi}{\partial\gamma^2} + \eta^2\,\frac{\partial^2\chi}{\partial\eta^2}\right], \quad 0 < \gamma,\eta < 1,,,1 < \tau \le 2, \beta > 0. \tag{55}$$

Subject to the BC:

$$\chi(0,\eta,\beta) = 0\;,\;\; \chi(1,\eta,\beta) = 4\cosh\beta$$
$$\chi(\gamma,0,\beta) = 0\;,\;\; \chi(\gamma,2\pi,\beta) = 4\sinh\beta \tag{56}$$

and IC:

$$\chi(\gamma,\eta,0) = \gamma^2,\;\; \chi_\beta(\gamma,\eta,0) = \eta^2 \tag{57}$$

By using ADM, Eq. (58) can be written in the form,

$$L_{\beta\chi} = \frac{1}{12}\left[\gamma^2\frac{\partial^2\chi}{\partial\gamma^2} + \eta^2\,\frac{\partial^2\chi}{\partial\eta^2}\right], \tag{58}$$

where $L = D^{\tau}_{\beta},\; 1 < \tau \le 2$.

Taking L^{-1}_{β} of Eq. (58), we find that:

$$\chi(\gamma,\eta,\beta) = \gamma^4 + \eta^4\beta + L^{-1}_{\beta}\left[\frac{1}{12}\left(\gamma^2\frac{\partial^2\chi}{\partial\gamma^2} + \eta^2\,\frac{\partial^2\chi}{\partial\eta^2}\right)\right].$$

Using the initial approximation, we find that:

$$\chi_0(\gamma,\eta,\beta) = \gamma^4 + \eta^4\beta. \tag{59}$$

By using the new technique of initial approximation χ^*_ρ, we have:

$$\chi_{\rho+1} = \frac{1}{12}L^{-1}_{\beta}\left[\gamma^2\left(\chi^*_\rho\right)_{\gamma\gamma} + \eta^2\left(\chi^*_\rho\right)_{\eta\eta}\right], \rho = 0,1,2,\ldots \tag{60}$$

By applying a new approximation χ^*_ρ, we have:

$$\chi^*_\rho(\gamma,\eta,\beta) = \chi_\rho(\gamma,\eta,\beta) + (1-\gamma)\left[\chi(0,\eta,\beta) - \chi_\rho(0,\eta,\beta)\right] + \gamma\left[\chi(1,\eta,\beta) - \chi_\rho(1,\eta,\beta)\right] + (1-\eta)\left[\chi(\gamma,0,\beta) - \chi_\rho(\gamma,0,\beta)\right] + \eta\left[\chi(\gamma,1,\beta) - \chi_\rho(\gamma,1,\beta)\right],\;\; \rho = 0,1,2,\ldots \tag{61}$$

Let $\rho = 0$; we then obtain:

$$\chi^*_0(\gamma,\eta,\beta) = \chi_0(\gamma,\eta,\beta) + (1-\gamma)[\chi(0,\eta,\beta) - \chi_0(0,\eta,\beta)] + \gamma[\chi(1,\eta,\beta) - \chi_0(1,\eta,\beta)] + (1-\eta)[\chi(\gamma,0,\beta) - \chi_0(\gamma,0,\beta)] + \eta[\chi(\gamma,1,\beta) - \chi_0(\gamma,1,\beta)] = \gamma^4 + \eta^4\beta\,.$$

Using Eq. (60), we obtain:

$$\begin{aligned}
\chi_1 &= \frac{1}{12}L^{-1}_{\beta}\left[\gamma^2(\chi^*_0)_{\gamma\gamma} + \eta^2(\chi^*_0)_{\eta\eta}\right] \\
&= \gamma^4\frac{\beta^{\tau}}{\Gamma(\tau+1)} + \eta^4\frac{\beta^{\tau+1}}{\Gamma(\tau+2)} \\
\chi_2 &= \frac{1}{12}L^{-1}_{\beta}\left[\gamma^2(\chi^*_1)_{\gamma\gamma} + \eta^2(\chi^*_1)_{\eta\eta}\right] \\
&= \gamma^4\frac{\beta^{2\tau}}{\Gamma(2\tau+1)} + \eta^4\frac{\beta^{2\tau+1}}{\Gamma(2\tau+2)} \\
\chi_3 &= \frac{1}{12}L^{-1}_{\beta}\left[\gamma^2(\chi^*_2)_{\gamma\gamma} + \eta^2(\chi^*_2)_{\eta\eta}\right] \\
&= \gamma^4\frac{\beta^{3\tau}}{\Gamma(3\tau+1)} + \eta^4\frac{\beta^{3\tau+1}}{\Gamma(3\tau+2)}
\end{aligned} \tag{62}$$

Thus, the new ADM solution for Eq. (55) can be written in a series form:

$$\begin{aligned}
\chi(\gamma,\eta,\beta) &= \chi_0(\gamma,\eta,\beta) + \chi_1(\gamma,\eta,\beta) + \chi_2(\gamma,\eta,\beta) + \chi_3(\gamma,\eta,\beta) + \cdots \\
&= \gamma^4\left[1 + \frac{\beta^{\tau}}{\Gamma(\tau+1)} + \frac{\beta^{2\tau}}{\Gamma(2\tau+1)} + \frac{\beta^{3\tau}}{\Gamma(3\tau+1)} + \cdots\right] \\
&\quad + \eta^4\left[\beta + \frac{\beta^{\tau+1}}{\Gamma(\tau+2)} + \frac{\beta^{\tau+3}}{\Gamma(2\tau+2)} + \frac{\beta^{\tau+5}}{\Gamma(3\tau+2)} + \cdots\right].
\end{aligned} \tag{63}$$

If $\tau = 2$, we obtain:

$$\begin{aligned}
\chi(\gamma,\eta,\beta) &= \gamma^4\left(1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \frac{\beta^6}{6!} + \cdots\right) + \\
&\quad \eta^4\left(\beta + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \frac{\beta^7}{7!} + \cdots\right) \\
&= \gamma^4\cosh\beta + \eta^4\sinh\beta
\end{aligned} \tag{64}$$

which is the exact solution of Eq. (55).

Example 5. Consider the following one-dimensional nonlinear heat – similar to Eq. (34):

$$D^{\tau}_{\beta}\chi = \gamma\chi\frac{\partial^2\chi}{\partial\gamma^2} - 8\gamma^3\,\frac{\beta^{2\tau+2}}{\Gamma(\tau+2)^2} + 2\gamma^2\beta^{\tau}, \quad 0 < \gamma,\tau \le 1\;,,\;,\beta > 0. \tag{65}$$

Subject to the BC:

$$\chi(0,\;\beta) = 0\;,\;\; \chi(1,\beta) = \frac{2\beta^{\tau+1}}{\Gamma(\tau+2)}\;\;,\;\; \beta > 0 \tag{66}$$

and IC:

$$\chi(\gamma,0) = 0 \tag{67}$$

By using ADM, Eq. (65) can be written in the form,

$$L_\chi = \gamma\chi\frac{\partial^2\chi}{\partial\gamma^2} - 8\gamma^3\,\frac{\beta^{2\tau+2}}{\Gamma(\tau+2)^2} + 2\gamma^2\beta^{\tau}\,, \tag{68}$$

where $L = D^{\tau}_{\beta},\; 0 < \tau \le 1$.

Taking L^{-1} of Eq. (68), we find that:

$$\chi(\gamma,\beta) = \chi(\gamma,0) + L^{-1}[2\gamma^2\beta^{\tau}] + L^{-1}\left[\gamma\chi\frac{\partial^2\chi}{\partial\gamma^2} - 8\gamma^3\,\frac{\beta^{2\tau+2}}{\Gamma(\tau+2)^2}\right]. \tag{69}$$

Using the initial approximation, we find that:

$$\chi_0(\gamma,\beta) = 2\gamma^2\,\frac{\beta^{2\tau}}{\Gamma(2\tau+1)}. \tag{70}$$

By using the new technique of initial approximation χ^*_ρ, we have:

$$\chi_{\rho+1} = L^{-1}\left[\gamma\chi^*_\rho\chi^*_{\rho\gamma\gamma} - 8\gamma^3\,\frac{\beta^{2\tau+2}}{\Gamma(\tau+2)^2}\right], \rho = 0,1,2,3,\ldots \tag{71}$$

By applying a new approximation χ_ρ^*, we have:

$$\chi_\rho^*(\gamma,\beta) = \chi_\rho(\gamma,\beta) + (1-\gamma)[0-\chi_\rho(0,\beta)] \\ +\gamma\left[\frac{2\beta^{\tau+1}}{\Gamma(\tau+2)} - \chi_\rho(1,\beta)\right], \quad \rho = 0,1,2,\ldots \tag{72}$$

Let $\rho = 0$; we then obtain:

$$\chi_0^*(\gamma,\beta) = \chi_0(\gamma,\beta) + (1-\gamma)[0-\chi_0(0,\beta)] \\ +\gamma\left[\frac{2\beta^{\tau+1}}{\Gamma(\tau+2)} - \chi_0(1,\beta)\right]$$

Using Eq. (71), we obtain:

$$\chi_1(\gamma,\beta) = 0\,, \tag{73}$$

Thus, the new ADM solution for Eq. (65) can be written in a series form:

$$\chi(\gamma,\beta) = \chi_0(\gamma,\beta) + \chi_1(\gamma,\beta), \\ = 2\gamma^2\frac{\beta^{2\tau}}{\Gamma(2\tau+1)} \tag{74}$$

Let $\rho = 1$; we then obtain:

$$\chi(\gamma,\beta) = \gamma^2\,\beta^2 \tag{75}$$

Example 6. Consider the following one-dimensional nonlinear wave – similar to Eq. (34):

$$D_\beta^\tau\chi = \gamma\chi\frac{\partial^2\chi}{\partial\gamma^2} - 8\gamma^3\,\frac{\beta^{2\tau}}{\Gamma(\tau+2)^2} + 2\gamma^2, \\ 1 < \gamma,\tau \le 2 \;,, \;,\beta > 0. \tag{76}$$

Subject to the BC:

$$\chi(0,\ \beta) = 0\;,\;\chi(1,\beta) = \frac{2\beta^\tau}{\Gamma(\tau+1)}\quad,\;\beta > 0 \tag{77}$$

and IC:

$$\chi(\gamma,0) = 0\;,\;\chi_\beta(\gamma,0) = 0 \tag{78}$$

By using ADM, Eq. (76) can be written in the form,

$$L_\chi = \gamma\chi\frac{\partial^2\chi}{\partial\gamma^2} - 8\gamma^3\,\frac{\beta^{2\tau}}{\Gamma(\tau+1)^2} + 2\gamma^2\,, \tag{79}$$

where $L = D_\beta^\tau,\ 1 < \tau \le 2\,.$

Taking L^{-1} of Eq. (79), we find that:

$$\chi(\gamma,\beta) = \chi(\gamma,0) + L^{-1}[2\gamma^2] \\ + L^{-1}\left[\gamma\chi\frac{\partial^2\chi}{\partial\gamma^2} - 8\gamma^3\,\frac{\beta^{2\tau}}{\Gamma(\tau+1)^2}\right]. \tag{80}$$

Using the initial approximation, we have:

$$\chi_0(\gamma,\beta) = 2\gamma^2\frac{\beta^\tau}{\Gamma(\tau+1)}. \tag{81}$$

By using the new technique of initial approximation χ_ρ^*, we have:

$$\chi_{\rho+1} = L^{-1}\left[\gamma\chi_\rho^*\chi_{\rho\gamma\gamma}^* - 8\gamma^3\,\frac{\beta^{2\tau}}{\Gamma(\tau+1)^2}\right], \rho = 0,1,2,3,\ldots \tag{82}$$

By applying a new approximation χ_ρ^*, we have:

$$\chi_\rho^*(\gamma,\beta) = \chi_\rho(\gamma,\beta) + (1-\gamma)[0-\chi_\rho(0,\beta)] \\ +\gamma\left[\frac{2\beta^\tau}{\Gamma(\tau+1)} - \chi_\rho(1,\beta)\right], \quad \rho = 0,1,2,\ldots \tag{83}$$

Let $\rho = 0$; we then obtain:

$$\chi_0^*(\gamma,\beta) = \chi_0(\gamma,\beta) + (1-\gamma)[0-\chi_0(0,\beta)] \\ +\gamma\left[\frac{2\beta^\tau}{\Gamma(\tau+1)} - \chi_0(1,\beta)\right]$$

Using Eq. (71), we obtain:

$$\chi_1(\gamma,\beta) = 0\,, \tag{84}$$

Thus, the new ADM solution for Eq. (76) can be written in a series form:

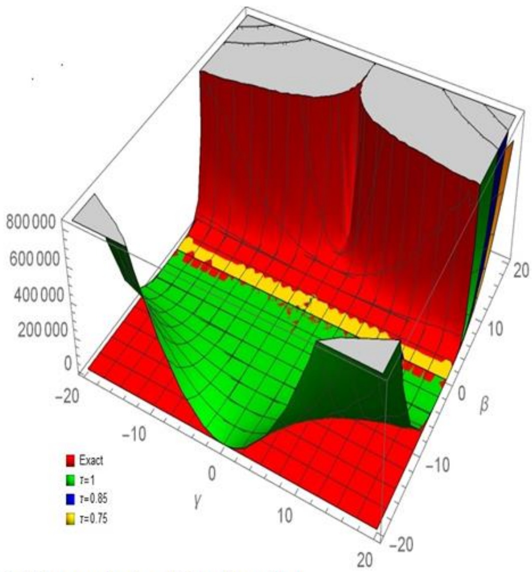
$$\chi(\gamma,\beta) = \chi_0(\gamma,\beta) + \chi_1(\gamma,\beta), \\ = 2\gamma^2\frac{\beta^\tau}{\Gamma(\tau+1)} \tag{85}$$

Let $\rho = 1$; we then obtain:

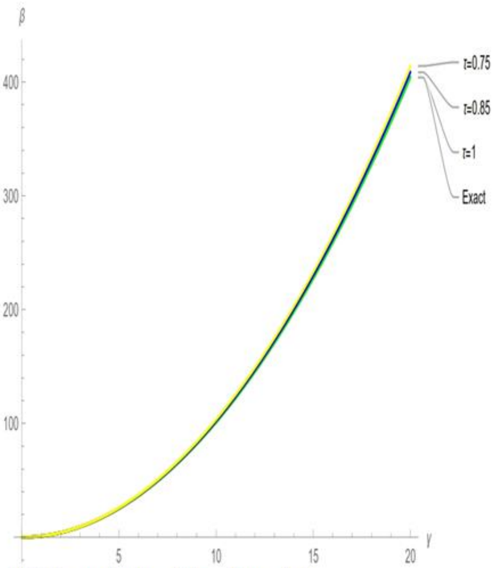
$$\chi(\gamma,\beta) = \gamma^2\,\beta^2 \tag{86}$$

5. NUMERICAL RESULT

The derived solutions from Example 1 are displayed at various fractional orders of the derivatives in Figs. 1(a) and 1(b).



(a) 3D approximate solution – Example 1



(b) 2D approximate solution – Example 1

Fig. 1. Comparison between exact and approximate solutions

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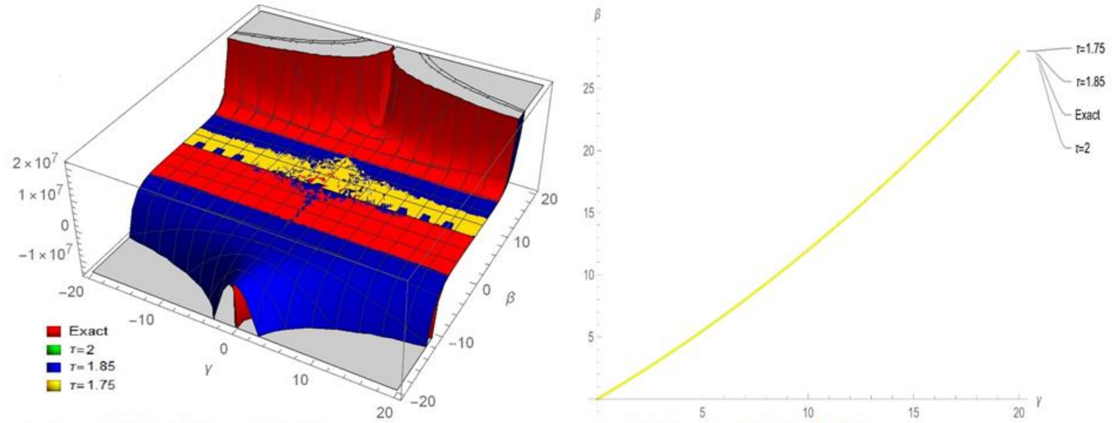
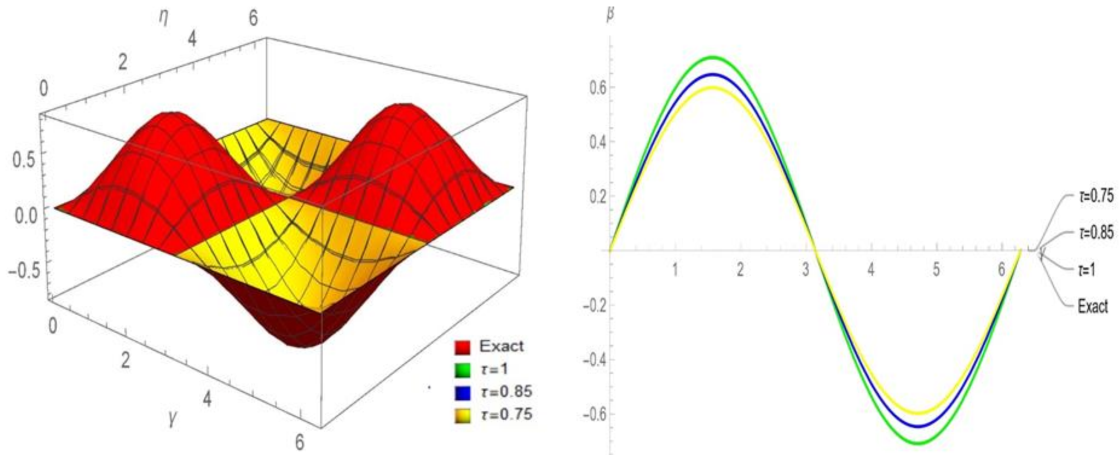


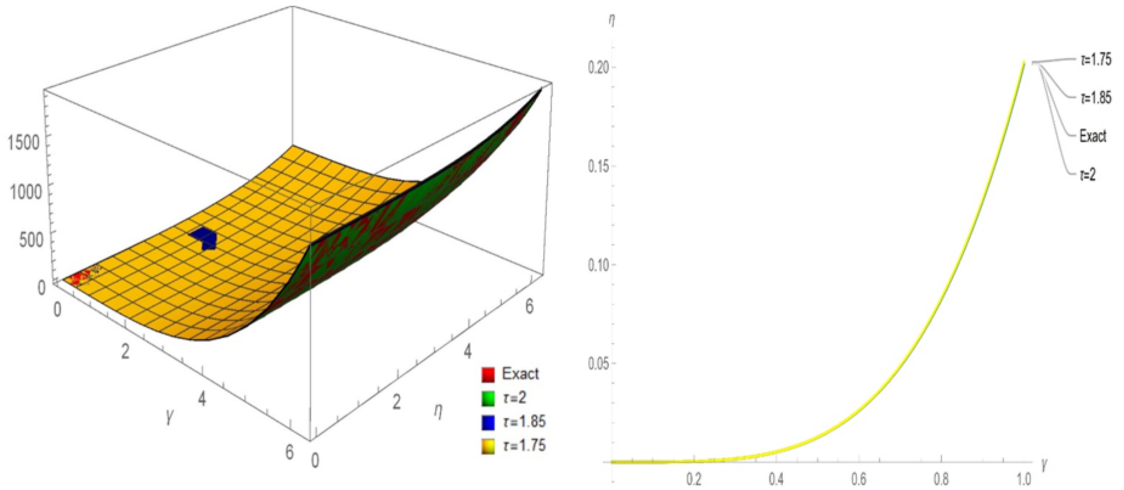
(a) 3D approximate solution – Example 2 (b) 2D approximate solution – Example 2

Fig. 2. Comparison between exact and approximate solutions



(a) 3D approximate solution – Example 3 (b) 2D approximate solution – Example 3

Fig. 3. Comparison between exact and approximate solutions



(a) 3D approximate solution – Example 4 (b) 2D approximate solution – Example 4

Fig. 4. Comparison between exact and approximate solutions

The graphs of the 2D and 3D ADM solutions in Figs. 1(a) and 1(b) corroborate the closed contact with the precise solution of Example 1 and show the 2D and 3D ADM solutions τ.The closed contact between Example 1's precise and found solutions at the integer derivative order is being investigated. The graphs show how closely related the exact and derived outcomes are. The proposed approach yields an accurate solution for Example 1 as a consequence. Figs. 2(a) and 2(b) show the obtained solutions from Example 2 at different fractional orders of the derivative solution for Example 1 as a consequence. Figs. 2(a) and 2(b) show the obtained solutions from Example 2 at different fractional orders of the derivatives. The closed contact with the exact solution of Example 2 is supported by the graphs of the 2D and 3D ADM solutions. As a consequence, using the provided technique, it can be ascertained that Example 2's solution is correct. Figs. 3(a) and 3(b) show the analytical and exact solutions for χ (γ,β) of Example 3 at β = 0.1,0.2 and 0 < γ ≤ 2π for various fractional-orders. The graphical behavior of the exact and analytical solutions to χ (γ,β) Figs. 4(a) and 4(b) show the analytical and exact solutions for χ (γ,β) of Example 4 at β= 0.2 and 0 < γ,≤ 1 for various fractional-orders. The graphical behaviour of the exact and analytical solutions to χ (γ,β).Tabs. 1–4 contrast the exact answers that have been presented with the absolute inaccuracy at various fractional orders. The findings in the figures and tables clearly show that our approach to finding a solution quickly converges on a correct response. Last but not least, the images and figures show that the suggested processes are more exact and quickly converge to accurate results.

Tab. 1. Solution for the (first three approximations) with exact solution, with mesh points $\beta = 0.01$, for Eq. (21)

$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\tau = 0.75}$	$\boldsymbol{\tau = 0.85}$	$\boldsymbol{\tau = 1}$	**Exact**	**Error** $\boldsymbol{\tau = 0.75}$	**Error** $\boldsymbol{\tau = 0.85}$	$\boldsymbol{Error\ \tau = 1}$
0.01	0	0	0	0	0	0	0	0
	0.1	0	0.010213602	0.010100502	0.010100502	0.000251223	0.000113101	8.34575×10^{-15}
	0.2	0.041406899	0.04085441	0.040402007	0.040402007	0.001004892	0.000452403	3.3383×10^{-14}
	0.3	0.093165522	0.091922422	0.090904515	0.090904515	0.002261007	0.001017907	7.51066×10^{-14}
	0.4	0.165627595	0.163417639	0.161608027	0.161608027	0.004019568	0.001809612	1.33532×10^{-13}
	0.5	0.258793117	0.255340061	0.252512542	0.252512542	0.006280575	0.002827519	2.08611×10^{-13}
	0.6	0.372662089	0.367689688	0.36361806	0.36361806	0.009044029	0.004071628	3.00426×10^{-13}
	0.7	0.50723451	0.50046652	0.494924582	0.494924582	0.012309928	0.005541938	4.0884×10^{-13}
	0.8	0.66251038	0.653670556	0.646432107	0.646432107	0.016078273	0.007238449	5.34128×10^{-13}
	0.9	0.8384897	0.827301798	0.818140635	0.818140635	0.020349064	0.009161162	$6.7590377 \times 10^{-13}$
	1	1.035172469	1.021360244	1.010050167	1.010050167	0.025122301	0.011310077	8.34443×10^{-13}

Tab. 2. Solution for the (first three approximations) with exact solution, with mesh points $\beta = 0.02$, for Eq. (30)

$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\tau = 1.75}$	$\boldsymbol{\tau = 1.85}$	$\boldsymbol{\tau = 2}$	**Exact**	**Error** $\boldsymbol{\tau = 1.75}$	**Error** $\boldsymbol{\tau = 1.85}$	$\boldsymbol{Error\ \tau = 2}$
0.02	0	0	0	0	0	0	0	0
	0.1	0.1002	0.1002	0.1002	0.1001	1.18937×10^{-7}	6.89018×10^{-8}	2.66667×10^{-8}
	0.2	0.200801	0.2008	0.2008	0.2004	4.75748×10^{-7}	2.75607×10^{-7}	1.06667×10^{-7}
	0.3	0.301801	0.301801	0.3018	0.3009	1.07043×10^{-6}	6.20116×10^{-7}	2.4×10^{-6}
	0.4	0.403202	0.403201	0.403201	0.4016	1.90299×10^{-6}	1.10243×10^{-6}	4.26667×10^{-7}
	0.5	0.505003	0.505002	0.505001	0.5025	2.97343×10^{-6}	1.72254×10^{-6}	6.66667×10^{-7}
	0.6	0.607205	0.607203	0.607201	0.6036	4.28173×10^{-6}	2.48046×10^{-6}	9.6×10^{-7}
	0.7	0.709806	0.709804	0.709802	0.7049	5.82792×10^{-6}	3.37619×10^{-6}	1.30667×10^{-6}
	0.8	0.812808	0.812805	0.812803	0.8064	7.61197×10^{-6}	4.40971×10^{-6}	1.70667×10^{-6}
	0.9	0.916211	0.916207	0.916203	0.9081	9.6339×10^{-6}	5.58104×10^{-6}	2.16×10^{-6}
	1	1.02001	1.02001	1.02	1.01	$1{,}18937 \times 10^{-5}$	6.89018×10^{-6}	2.66667×10^{-6}

Tab. 3. Solution for the (first three approximations) with exact solution, with mesh points $\beta = 0.02$, for Eq. (30)

$\boldsymbol{\beta}$	$\boldsymbol{\eta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\tau = 0.75}$	$\boldsymbol{\tau = 0.85}$	$\boldsymbol{\tau = 1}$	**Exact**	**Error** $\boldsymbol{\tau = 0.75}$	**Error** $\boldsymbol{\tau = 0.85}$	**Error** $\boldsymbol{\tau = 1}$
0.1	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	0.517896942	0.559870154	0.614	0.614048065	0.096151123	0.05417791	4.80648E-05
		$\frac{4\pi}{3}$	–0.517896942	–0.559870154	–0.614	–0.614048065	0.096151123	0.05417791	4.80648E-05
0.2		$\frac{2\pi}{3}$	0.400780878	0.443831887	0.502	0.502740035	0.101959157	0.058908148	0.000740035
		$\frac{4\pi}{3}$	–0.400780878	–0.443831887	–0.502	–0.502740035	0.101959157	0.058908148	0.000740035

Tab. 4. Solution for the (first three approximations) with exact solution, with mesh points $\beta = 0.2$, for Eq. (55)

$\boldsymbol{\beta}$	$\boldsymbol{\eta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\tau = 1.75}$	$\boldsymbol{\tau = 1.85}$	$\boldsymbol{\tau = 2}$	**Exact**	**Error** $\boldsymbol{\tau = 1.75}$	**Error** $\boldsymbol{\tau = 1.85}$	**Error** $\boldsymbol{\tau = 2 = 2}$
0.2	0.1	0.25	0.000791879	0.000789268	0.000786479	0.000786479	5.40013×10^{-6}	2.78865×10^{-6}	5.51412×10^{-15}
		0.5	0.01266991	0.012628127	0.01258351	0.01258351	0.0000864	0.0000446	8.82159×10^{-14}
		0.75	0.064141377	0.063929852	0.06370398	0.06370398	0.000437397	0.000225873	4.46601×10^{-13}
	0.05	0.25	0.000798353	0.00079569	0.000792844	0.000792844	5.5089×10^{-6}	2.84615×10^{-6}	5.91053×10^{-6}
		0.5	0.012676384	0.01263455	0.012589876	0.012589876	0.0000865	0.0000447	8.86131×10^{-14}
		0.75	0.064147851	0.063936275	0.063710345	0.063710345	0.000437506	0.00022593	4.47004×10^{-14}

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6. CONCLUSION

Fractional-order heat-like and wave-like equations with initial and boundary conditions are studied analytically in this paper. A novel approach based on ADM is provided for the solution of specified problems in a very easy and effective manner. For each case, fractional derivatives are defined in the Caputo sense. The approach is particularly well-suited to solving fractional PDEs with beginning and boundary conditions. Additionally, information demonstrating the output of this approach is presented in the form of graphs and tables to highlight the current technique's best applicability. The results also show that the techniques are a very effective, useful and accurate way to solve heat and wave equations with initial and boundary conditions.

REFERENCES

1. Eltaib M Abd Elmohmoud, Mohamed Z Mohamed . Numerical treatment of some fractional nonlinear equations by Elzaki transform.Journal of Taibah University for Science.2022;16(1): 774-787.
2. Metzler, R Nonnenmacher TF. Space-and time-fractional diffusion and wave equations fractional Fokker–Planck equations and physical motivation. Chem. Phys.2002;284:67–90.
3. Metzler, R Klafter J. The random walk's guide to anomalous diffusion A fractional dynamics approach. Phys Rep.2000; 339: 1–77.
4. M Subramanian, M Manigandan , C Tunç, T N Gopal , J Alzabut . On system of nonlinear coupled differential equations and inclusions involving Caputo-type sequential derivatives of fractional order. Journal of Taibah University for Science.2022;16(1):1-23.
5. Imran Talib, Ali Raza, Abdon Atangana , Muhammad Bilal Riaz . Numerical study of multi- order fractional differential equations with constant and variable coefficients. Journal of Taibah University for Science.2022;16(1):608-620.
6. Jothimani, K Kaliraj, K Hammouch, Z Ravichandran C. New results on controllability in the framework of fractional integro differential equations with nondense domain. Eur Phys J Plus . 2019;134(9):441.
7. Khan H, Shah R, Kumam P, Arif M. Analytical Solutions of Fractional-Order Heat and Wave Equations by the Natural Transform Decomposition Method. Entropy. 2019; 21: 597.
8. Lei Y, Wang H, Chen X,Yang X, You Z, Dong S, Gao J. Shear property high-temperature rheological performance and low-temperature flexibility of asphalt mastics modified with bio-oil. Constr Build Mater.2018;174:30–37.
9. Zhang ZY. Symmetry determination and nonlinearization of a nonlinear time-fractional partial differential equation.Proc R Soc.2020;476:2019-0564.
10. Mohamed Z Mohamed, Amjad E Hamza , Abdelilah Kamal H Sedeeg .An Efficient Approximate Solutions of the fractional coupled Burger's equation by conformable double Sumudu transforms. Ain shams journal.2022;16: 101-879.
11. Shah R, Khan H, Arif M, et al. Application of Laplace–Adomian decomposition method for the analytical solution of third-order dispersive fractional partial differential equations. Entropy . 2019;21(4):335.
12. Shah HF, ur Rahman K, Shahzad M G. Numerical solution of fractional order smoking model via Laplace Adomian decomposition method. Alex Eng J. 2018;57(2):1061–1069.
13. Rubbab Q, Nazeer M, Ahmad F, et al. Numerical simulation of advection–diffusion equation with Caputo Fabrizio time fractional derivative in cylindrical domains applications of pseudo-spectral collocation method. Alexandria Eng J. 2021;60(1):1731–1738.
14. Agbata B C, Shior MM, Olorunni shola OA, Ezugorie IG, Obeng-Denteh. Analysis of Homotopy Perturbation Method (HPM) and its Application for Solving Infectious Disease Models. IJMSS 2021,9: 27-38.
15. Hamoud A, Dawood L, Ghadle K, Atshan S. Usage of the modified variational iteration technique for solving Fredholm integro-differential equations. Int J Mech Prod Eng Res Dev.2019; 9(2):895–902.
16. Osman M, Gong ZT, Mohammed A. Differential transform method for solving fuzzy fractional wave equation. J Comput Anal Appl.2021; 29(3):431–453.
17. Omer Acan1, Omer Firat , Yildiray Keskin, Galip Oturanc. Conformable variational iteration method. NTMSCI.2017;5(1):172-178.
18. G Adomian. A review of the decomposition method in applied mathematics. J Math Anal Appl.1988;135:501–544.
19. G. Adomian Solving Frontier Problems of Physics The Decomposition method. Kluwer Academic Publishers Boston. 1994.
20. S Momani, Z Odibat. Numerical comparison of methods for solving linear differential equations of fractional order. Chaos Solitons Fractals. 2007;31(5):1248–1255.
21. Z Odibat, S Momani. Approximate solutions for boundary value problems of time-fractional wave equation. Appl Math Comput.2006;181(1):767–774.
22. V Marinca. An approximate solution for one-dimensional weakly nonlinear oscillations. J Nonlinear Sci Numer Simul.2002;3(2): 107–110.
23. TH Hao. Search for variational principles in electrodynamics by Lagrange method. Int J Nonlinear Sci Numer Simul.2005;6(2): 209–210.
24. Mustafa S. Hajira, Khan H, Shah R, Masood S. A Novel Analytical Approach for the Solution of Fractional-Order Diffusion-Wave Equations. Fractal Fract .2021;5:206.
25. Mainardi F. Fractional Calculus and Waves in Linear Viscoelasticity An Introduction to Mathematical Models. World Scientific Singapore. 2010.
26. Anh VV, Leonenko NN. Harmonic analysis of random fractional diffusion–wave equations. Appl Math Comput.2003;141:77–85.
27. Ali EJ. A new technique of initial boundary value problems using Adomian decomposition method. Int Math Forum .2012;7:799–814.
28. Mohamed Z Mohamed , Tarig M Elzaki .Applications of new integral transform for linear and nonlinear fractional partial differential equations.Journal of King Saud University – Science. 2020;32:544–549.
29. Mustafa S, Hajira, Khan Hm Shah R, Masood S. A Novel Analytical Approach for the Solution of Fractional-Order Diffusion-Wave Equations. Fractal Fract. 2021;5:206.
30. Alia A, Shaha K, Lib Y, Khana RA. Numerical treatment of coupled system of fractional order partial differential equations. J Math Comput Sci. 2019;19:74–85.
31. Hassan Khan , Rasool Shah , Poom Kumam , and Muhammad Arif. Analytical Solutions of Fractional-Order Heat and Wave Equations by the Natural Transform Decomposition Method. mdpi journal entropy.2019;21:597.
32. S Sarwar, Salem Alkhalaf, S Iqbal, MA Zahid. A note on optimal homotopy asymptotic method for the solutions of fractional order heat- and wave-like partial differential equations. Computers & Mathematics with Applications. 2020;70(5):942-953.
33. Shah R, Khan H, Mustafa S, Kumam P, Arif M. Analytical Solutions of Fractional-Order Diffusion Equations by Natural Transform Decomposition Method. Entropy,2019, 557.
34. M A AL-Jawary. An Efficient Treatments For Linear And Nonlinear Heat-Like And Wave-Like Equations With Variable Coefficients. IOSR Journal of Mathematics.2015;11(4):01-13.

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