

APPROXIMATE SOLUTION OF PAINLEVÉ EQUATION I BY NATURAL DECOMPOSITION METHOD AND LAPLACE DECOMPOSITION METHOD

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Abstract: The Painlevé equations and their solutions occur in some areas of theoretical physics, pure and applied mathematics. This paper applies natural decomposition method (NDM) and Laplace decomposition method (LDM) to solve the second-order Painlevé equation. These methods are based on the Adomian polynomial to find the non-linear term in the differential equation. The approximate solution of Painlevé equations is determined in the series form, and recursive relation is used to calculate the remaining components. The results are compared with the existing numerical solutions in the literature to demonstrate the efficiency and validity of the proposed methods. Using these methods, we can properly handle a class of non-linear partial differential equations (NLPDEs) simply.

Novelty: One of the key novelties of the Painlevé equations is their remarkable property of having only movable singularities, which means that their solutions do not have any singularities that are fixed in position. This property makes the Painlevé equations particularly useful in the study of non-linear systems, as it allows for the construction of exact solutions in certain cases. Another important feature of the Painlevé equations is their appearance in diverse fields such as statistical mechanics, random matrix theory and soliton theory. This has led to a wide range of applications, including the study of random processes, the dynamics of fluids and the behaviour of non-linear waves.

Keywords: natural decomposition method, Laplace decomposition method, series solution, Adomian polynomial, Painlevé equation

1. INTRODUCTION

The special functions are vital in studying linear differential equations (LDEs) and are significant in mathematical physics. These functions include the Bessel, Whittaker, parabolic cylindrical, Airy and hypergeometric functions. A few of them have the same names as those functions because they are solutions to LDEs with rational coefficients. The Bessel functions are solutions of the Bessel equation, the second-order LODEs with single irregular singularity, and are utilised to explain the motion of planets and artificial satellites by employing the Kepler equation. The Painlevé equations were first discovered by Paul Painlevé and Gambier more than 100 years ago [1]. Painlevé investigated many second-order equations and categorised the singularities form. Painlevé [2] and some of his collaborators derived the Painlevé equations while investigating the non-linear partial differential equations (NLPDEs).

Consider the second-order non-linear equation of the form:

$$\left(\frac{d^2y}{dt^2}\right) = R\left(t, v, \frac{dv}{dt}\right),$$

where $R\left(t, v, \frac{dv}{dt}\right)$ is rational in v and v' and function is analytic in t .

The only movable singularity in these differential equations is a pole known as the Painlevé property. This significant property is expressed for all linear equations, but non-linear equations rarely have it. Painlevé and Gambier focused on the arrangement of the

singularity structure with the polynomial coefficient of the second-order differential equation. They established that subject to a certain transformation; any such equations can be transformed into one of the 50 canonical forms, which is a significant achievement. The question arose: Which of the following equations are irreducible and need to specify new transcendental functions? Painlevé described that 44 of these 50 equations are reducible, except only these six equations that require a new function for their solution, which are named Painlevé equations. Moreover, some experts believe that Painlevé functions will arise as a unique member of the special function in the 21st century. Many mathematicians have used Painlevé equations to explain the growing nature of the systems since their discovery. Painlevé equations play a vital role in many physical problems such as the asymptotic behaviour of the non-linear equation [3], superconductivity [4], negative curvature surfaces [5], Bose–Einstein condensation, Stokes phenomena [6], hyperasymptotics solution [7] and non-linear optics [8]. Furthermore, some NLPDEs, including the Bousinesq and Korteweg–de Vries (Kdv) equations, may be expressed in terms of the Painlevé transcendent [9, 10].

These six equations have a wide range of remarkable characteristics and applications. The Painlevé equations were initially discovered from primarily mathematical considerations, but more recently, they have been used in several significant physical applications, such as statistical mechanics, plasma physics, non-linear waves, quantum gravity, quantum field theory, general relativity, non-linear optics and fibre optics [11]. The Painlevé equations have also attracted significant interest because they are

reductions of the soliton equations that can be solved by inverse scattering [12] transform, such as the Kdv, the modified Kortewegde Vries equation (MKdv), the cylindrical Kortewegde Vries equation, the equations of the Boussinesq and Kadomtsev–Petviashvili type, the non-linear Schrodinger equation and the sine-Gordon equation. Connections are briefly described in [13–15]. Moreover, the Painlevé equations can be represented as equations of isomonodromic deformations of auxiliary linear systems of differential equations [16].

In recent years, a wide range of numerical methods has been developed to solve the second-order Painlevé differential equation, such as homotopy perturbation method (HPM) [17], sinc-collocation method [18], optimisation method [19], variational iteration algorithm-I [20] and quasilinearisation method (QM) [21].

In this paper, the solution of the second-order Painlevé equation is obtained by two techniques Laplace decomposition method (LDM) and the natural decomposition method (NDM). Laplace transformation is the famous integral transform available to every researcher and can be used to solve linear ordinary, partial and integral equations in the time domain. Khan and his co-researcher [22] derived a new integral transform called N-transform; in 2012, it was renamed the natural transform [23]. The natural transform resembles Laplace [24] and Sumudu transforms [25]. Natural decomposition is used to solve a large number of the NLPDEs such as the sine-Gordon equation [26], time-fractional coupled KdV equation [27], gas dynamics equation [28] and inviscid burger equation [29].

There are several avenues of future work on the Painlevé I equation that could lead to new insights into its behaviour and applications in physics and mathematics.

One area of research is the study of its solutions and their properties. While the Painlevé I equation is known to have special types of solutions, such as the Painlevé transcendents, there is still much to be understood about their behaviour and how they can be used to describe physical systems. In particular, the study of the Painlevé I equation in the context of complex analysis and differential geometry could lead to new insights into its solutions and their properties.

Another area of research is the connection between the Painlevé I equation and other mathematical and physical systems. For example, recent work has shown that the Painlevé I equation arises in the study of the stability of certain non-linear waves in fluids. Investigating these connections could lead to new applications of the Painlevé I equation in other areas of physics and engineering. Finally, there is ongoing work on the numerical and computational aspects of the Painlevé I equation. While exact solutions can be constructed using certain methods, it is often necessary to use numerical techniques to study the behaviour of the equation in more complex settings. Developing efficient and accurate numerical methods for the Painlevé I equation could help to expand its use in a wide range of applications in physics and engineering. Overall, future work on the Painlevé I equation is likely to involve a combination of theoretical and computational approaches, with the goal of understanding its behaviour and applications in a wide range of contexts.

The paper is structured as follows. In Section 2, methodologies and its implementations of proposed techniques are briefly discussed. In Section 3, we apply NDM and LDM to the Painlevé equation to show the efficiency and accuracy of our methods. In Section 4, the important results related to the Painlevé equations are explained briefly. Section 5 ends this paper with the conclusions.

2. METHODOLOGY

In this section, we will discuss the methodologies of two proposed methods and also discuss how these methods handle the NLPDEs.

2.1. LDM

The general form of the NLPDE is given as follows with conditions:

$$Lv(x, t) + Rv(x, t) + Nv(x, t) = f(x, t) \quad (2.1.1)$$

(with initial conditions:

$$v(x, 0) = g(x), \quad v_t(x, 0) = h(x),$$

(where L denotes the linear differential operator of the second order:

$$L = \frac{\partial^2}{\partial t^2}, \quad (2.1.2)$$

where R represents the remaining less order linear operator, Nv is the non-linear operator and $f(x, t)$ is the non-homogeneous term.

Applying Laplace transformation of Eq. (2.1.1) with initial conditions:

$$\mathcal{L}[Lv(x, t)] + \mathcal{L}[Rv(x, t)] + \mathcal{L}[Nv(x, t)] = \mathcal{L}[f(x, t)] \quad (2.1.3)$$

Using the differentiation property of the Laplace transform, we have

$$\mathcal{L}[Lv(x, t)] = \frac{1}{s}g(x) + \frac{1}{s^2}h(x) - \frac{1}{s^2}\mathcal{L}[Rv(x, t) + Nv(x, t)]. \quad (2.1.4)$$

In the next step, the solution of Laplace transform in the form of infinite series is given as follows:

$$v(x, t) = \sum_{n=0}^{\infty} v_n(x, t). \quad (2.1.5)$$

So that the non-linear operator is decomposed as

$$Nv(x, t) = \sum_{n=0}^{\infty} A_n, \quad (2.1.6)$$

where A_n denotes the Adomian polynomials of the terms $v_0, v_1, v_2, \dots, v_n$ and can be calculated using this formula:

Substituting Eqs (2.1.5) and (2.1.6) into Eq. (2.1.4), we will get

$$A_n = \frac{1}{n!} \frac{d^n}{dx^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i v_i(x, t) \right) \right], \quad n = 0, 1, 2, \dots \quad (2.1.7)$$

$$\mathcal{L} \left[\sum_{n=0}^{\infty} v_n(x, t) \right] = \frac{g(x)}{s} + \frac{h(x)}{s^2} + \frac{1}{s^2} \mathcal{L}[f(x, t)] - \frac{1}{s^2} \mathcal{L}[Rv(x, t)] \quad (2.1.8)$$

Using the linearity property to the Laplace transform,

$$\sum_{n=0}^{\infty} \mathcal{L}[v_n(x, t)] = \frac{g(x)}{s} + \frac{h(x)}{s^2} + \frac{1}{s^2} \mathcal{L}[f(x, t)]$$

$$-\frac{1}{s^2} \mathcal{L}[Rv(x, t)] - \frac{1}{s^2} [\sum_{n=0}^{\infty} \mathcal{L}[A_n]] \quad (2.1.9)$$

Now, comparing Eq. (2.1.9) on both sides, we have

$$\mathcal{L}[v_0(x, t)] = \frac{g(x)}{s} + \frac{h(x)}{s^2} + \frac{1}{s^2} \mathcal{L}[f(x, t)] = H(x, s), \quad (2.1.10)$$

$$\mathcal{L}[v_1(x, t)] = -\frac{1}{s^2} \mathcal{L}[Rv_0(x, t)] + A_0, \quad (2.1.11)$$

$$\mathcal{L}[v_2(x, t)] = -\frac{1}{s^2} \mathcal{L}[Rv_1(x, t)] + A_1. \quad (2.1.12)$$

In general, we have

$$\mathcal{L}[v_{n+1}(x, t)] = -\frac{1}{s^2} \mathcal{L}[Rv_n(x, t)] + A_n \quad (2.1.13)$$

Applying the inverse Laplace transform to Eqs (2.1.10) – (2.1.13), the required recurrence relation is given as follows:

$$v_0(x, t) = H(x, t),$$

$$v_{n+1}(x, t) = -\mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L}[Rv_n(x, t)] + A_n \right], \quad n \geq 0.$$

$H(x, t)$ Specifies specifies the initial conditions for the term occurring from the source term.

By taking the Laplace transform of Eq. (2.1.13) and applying the Laplace inverse transform, we obtain the remaining terms of $v_0, v_1, v_2 \dots \dots v_n$ recursively.

2.2. Analysis of the NDM

The general form of the NDM is given below as follows:

$$Lv(x, t) + Mv(x, t) + Nv(x, t) = h(x, t), \quad (2.2.1)$$

with initial conditions:

$$v(x, 0) = f(x), \quad v_t(x, 0) = g(x). \quad (2.2.2)$$

Suppose L is the differential operator of the second order:

$$L = \frac{\partial^2}{\partial t^2}.$$

M is the less order remainder operator, Nv denotes the non-linear operator and $h(x, t)$ represents the source term.

Now, taking the N-transform of Eq. (2.2.1) on both sides,

$$N^+[Lv(x, t)] + N^+[Mv(x, t)] + N^+[Nv(x, t)] = N^+[f(x, t)]. \quad (2.2.3)$$

Using the N-transform properties, we have

$$\frac{s^2}{v^2} R(s, v) - \frac{s}{v^2} v(x, 0) - \frac{1}{v} v'(x, 0) + N^+[Mv(x, t)] + N^+[Nv(x, t)] = N^+[f(x, t)]. \quad (2.2.4)$$

Substitute Eq.(2.2.2) into Eq.(2.2.4), we obtain

$$R(s, v) = \frac{f(x)}{u} + \frac{vg(x)}{s^2} + \frac{v^2}{s^2} N^+[h(x, t)] - \frac{v^2}{s^2} N^+[Mv(x, t) + Nv(x, t)]. \quad (2.2.5)$$

Now, applying the inverse of the natural transform of Eq.(2.2.5), we get

$$v(x, t) = E(x, t) - N^- \left[\frac{v^2}{s^2} N^+[Mv(x, t) + Nv(x, t)] \right] \quad (2.2.6)$$

where $E(x, t)$ illustrates the term developed from initial conditions and source term.

The next step is that we present the solution of non-linear terms in the form of infinite series:

$$v(x, t) = \sum_{n=0}^{\infty} v_n(x, t). \quad (2.2.7)$$

So that the non-linear operator is decomposed as

$$Nv(x, t) = \sum_{n=0}^{\infty} A_n, \quad (2.2.8)$$

where A_n denotes the Adomain polynomials of $v_0, v_1, v_2 \dots \dots v_n$ and can be determined by this formula:

$$A_n = \frac{1}{n!} \frac{d^n}{dx^n} [N(\sum_{i=0}^{\infty} \lambda^i v_i(x, t))], \quad = 0, 1, 2, \dots \quad (2.2.9)$$

Substituting Eqs (2.2.7) and (2.2.8) into Eq. (2.2.6), we will get

$$\sum_{n=0}^{\infty} v_n(x, t) = E(x, t) - N^- \left[\frac{v^2}{s^2} N^+[M \sum_{n=0}^{\infty} v_n(x, t) + N \sum_{n=0}^{\infty} A_n(x, t)] \right]. \quad (2.2.10)$$

Now, comparing Eq. (2.2.10) on both sides, we get

$$v_0(x, t) = E(x, t),$$

$$v_1(x, t) = -N^- \left[\frac{v^2}{s^2} N^+[Mv_0(x, t) + A_0] \right],$$

$$v_2(x, t) = -N^- \left[\frac{v^2}{s^2} N^+[Mv_1(x, t) + A_1] \right].$$

In general, we have

$$v_{n+1}(x, t) = -N^- \left[\frac{v^2}{s^2} N^+[Mv_n(x, t) + A_n] \right], \quad n \geq 0. \quad (2.2.11)$$

By using the general recurrence relation given in Eq. (2.2.11), we determine easily the remaining terms of $v(x, t)$ as $v_0, v_1, v_2 \dots \dots v_n$ where v_0 is given the initial condition.

3. APPLICATIONS

In this section, we show the efficiency and high accuracy of the NDM and LDM for the solution of Painlevé equation I.

3.1. NDM for Painlevé equation

The second-order Painlevé equation [20,32] can be formulated as

$$v''(t) = 6v^2 + t, \quad (3.1.1)$$

subject to the initial conditions:

$$v(0) = 0, \quad v'(0) = 1. \quad (3.1.2)$$

By applying N-transform on both sides of Eq. (3.1.1),

$$N^+[v''(t)] = N^+[6v^2 + t]. \quad (3.1.3)$$

Using the properties of N-transform, we obtain

$$\frac{s^2}{v^2} R(s, v) - \frac{k}{v^2} v(0) - \frac{1}{v} v'(0) = N^+[6v^2] + \frac{v}{s^2}. \quad (3.1.4)$$

Substituting Eq. (3.1.2) into Eq. (3.1.2), we have

$$R(s, v) = \frac{v}{s^2} + \frac{v^3}{s^4} + \frac{v^2}{s^2} N^+[6v^2]. \quad (3.1.5)$$

Now, applying the inverse of N-transform, we have

$$N^{-1}[R(s, v)] = N^{-1}\left[\frac{v}{s^2} + \frac{v^3}{s^4} + \frac{v^2}{s^2} N^+[6v^2]\right] \quad (3.1.6)$$

Eq. (3.1.6) becomes

$$v(x, t) = t + \frac{t^3}{3!} + N^{-1}\left[\frac{v^2}{s^2} N^+[6v^2]\right]. \quad (3.1.7)$$

Eq. (3.1.7) can be written as

$$v(x, t) = t + \frac{t^3}{3!} + N^{-1}\left[\frac{v^2}{s^2} N^+[\sum_{n=0}^{\infty} A_n]\right]. \quad (3.1.8)$$

From Eq. (3.1.8), we can conclude that

$$\begin{aligned} v_0(x, t) &= t + \frac{t^3}{3!}, \\ v_1(x, t) &= N^{-1}\left[\frac{v^2}{s^2} N^+[A_0]\right], \\ v_2(x, t) &= N^{-1}\left[\frac{v^2}{s^2} N^+[A_1]\right], \\ v_{n+1}(x, t) &= N^{-1}\left[\frac{v^2}{s^2} N^+[A_n]\right], \quad n \geq 1. \end{aligned} \quad (3.1.9)$$

Therefore, from Eq. (3.1.9), the remaining terms of function $v(x, t)$ can easily be computed as follows:

$$\begin{aligned} v_1(x, t) &= \frac{1}{2}t^4 + \frac{1}{15}t^6 + \frac{1}{336}t^8, \\ v_2(x, t) &= \frac{1}{7}t^7 + \frac{1}{40}t^9 + \frac{71}{46200}t^{11} + \frac{1}{26208}t^{13}, \\ v_3(x, t) &= \frac{1}{28}t^{10} + \frac{23}{3080}t^{12} + \frac{5219}{8408400}t^{14} + \frac{3551}{144144000}t^{16} + \frac{95}{224550144}t^{18}, \\ v_4(x, t) &= \frac{3}{364}t^{13} + \frac{131}{64680}t^{15} + \frac{19867}{95295200}t^{17} + \frac{163469}{14378364000}t^{19} + \frac{163451}{491203440000}t^{21} + \frac{131}{7101398304}t^{23}. \end{aligned}$$

In this way, we obtained the approximate result of the function $v(x, t)$, which is given as

$$v(x, t) = \sum_{n=0}^{\infty} v_n(x, t), \quad (3.1.10)$$

$$\begin{aligned} v(x, t) &= t + \frac{t^3}{3!} + \frac{1}{2}t^4 + \frac{1}{15}t^6 + \frac{1}{336}t^8 + \frac{1}{7}t^7 + \frac{1}{40}t^9 + \frac{71}{46200}t^{11} + \frac{1}{26208}t^{13} + \frac{1}{28}t^{10} + \frac{23}{3080}t^{12} + \frac{5219}{8408400}t^{14} + \frac{3551}{144144000}t^{16} + \frac{95}{224550144}t^{18} + \frac{3}{364}t^{13} + \frac{131}{64680}t^{15} + \frac{19867}{95295200}t^{17} + \frac{163469}{14378364000}t^{19} + \frac{163451}{491203440000}t^{21} + \frac{131}{7101398304}t^{23} + \dots \end{aligned}$$

3.2. LDM for Painlevé equation

Consider the general form of the Painlevé equation:

$$v''(t) = 6v^2 + t, \quad (3.2.1)$$

subject to the initial conditions:

$$v(0) = 0, \quad v'(0) = 1. \quad (3.2.2)$$

Now, applying Laplace transform on both sides of Eq. (3.2.1)

with initial conditions:

$$\mathcal{L}[v''] = \mathcal{L}[6v^2] + \mathcal{L}[t],$$

$$s^2 v(x, s) + v(0)s - v'(0) = \mathcal{L}[6v^2] + \mathcal{L}[t]. \quad (3.2.3)$$

Substituting Eq. (3.2.2) into Eq. (3.2.3), we have

$$v(x, s) = \frac{1}{s^2} + \frac{1}{s^4} + \mathcal{L}[6v^2]. \quad (3.2.4)$$

Now, taking the inverse of the Laplace transform, we obtain

$$\mathcal{L}^{-1}[v(x, s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2} + \frac{1}{s^4} + \frac{1}{s^2} \mathcal{L}[6v^2]\right]. \quad (3.2.5)$$

Eq. (3.2.5) becomes

$$v(x, t) = t + \frac{t^3}{3!} + \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[6v^2]\right]. \quad (3.2.6)$$

Eq. (3.2.6) can be written as

$$v(x, t) = t + \frac{t^3}{3!} + \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[\sum_{n=0}^{\infty} A_n]\right]. \quad (3.2.7)$$

From Eq. (3.2.7), we can conclude that

$$\begin{aligned} v_0(x, t) &= t + \frac{t^3}{3!}, \\ v_1(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[A_0]\right], \\ v_2(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[A_1]\right], \\ v_3(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[A_2]\right]. \end{aligned}$$

Eventually,

$$v_{n+1}(x, t) = \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[A_n]\right], \quad n \geq 0. \quad (3.2.8)$$

Therefore, from Eq. (3.2.8), the other remaining terms of the function $u(x, t)$ can easily be computed as follows:

$$\begin{aligned} v_1(x, t) &= \frac{840t^4 + 112t^6 + 5t^8}{1680}, \\ v_2(x, t) &= \frac{1029600t^7 + 180180t^9 + 11076t^{11} + 275t^{13}}{7207200}, \\ v_3(x, t) &= \frac{11027016000t^{10} + 2305648800t^{12} + 191641680t^{14} + 7606242t^{16} + 130625t^{18}}{308756448000}, \end{aligned}$$

$$\begin{aligned} v_4(x, t) &= \frac{19460562660000t^{13}}{19460562660000t^{13}} + \frac{4782299886000t^{15}}{19460562660000t^{13}} + \frac{492262539300t^{17}}{19460562660000t^{13}} + \frac{26844879180t^{19}}{785708957t^{21}} + \frac{19460562660000t^{13}}{19460562660000t^{13}} + \frac{10307500t^{23}}{19460562660000t^{13}} + \dots \end{aligned}$$

$$\begin{aligned} v(x, t) &= t + \frac{t^3}{3!} + \frac{840t^4 + 112t^6 + 5t^8}{1680} + \frac{1029600t^7 + 180180t^9 + 11076t^{11} + 275t^{13}}{7207200} + \frac{11027016000t^{10}}{308756448000} + \frac{2305648800t^{12}}{308756448000} + \frac{191641680t^{14}}{308756448000} + \frac{7606242t^{16}}{308756448000} + \frac{130625t^{18}}{130625t^{18}} + \frac{19460562660000t^{13}}{19460562660000t^{13}} + \frac{4782299886000t^{15}}{4782299886000t^{15}} + \frac{308756448000}{4922625393000t^{17}} + \frac{26844879180t^{19}}{26844879180t^{19}} + \frac{2361214936080000}{2361214936080000} + \frac{2361214936080000}{2361214936080000} + \frac{2361214936080000}{2361214936080000} + \frac{10307500t^{23}}{2361214936080000} \end{aligned}$$

$$v(x, t) = \sum_{n=0}^{\infty} v_n(x, t), \quad (3.2.9)$$

Tab. 1. Comparison of the approximate solution of Painlevé equation I

t	MADM [30]	VIM [31]	LDM	NDM
0.1	0.1002601271	0.1002167477	0.1002167476	0.1002167476
0.2	0.2021288956	0.2021394527	0.2021394527	0.2021394527
0.3	0.3086306987	0.3086307490	0.3086307491	0.3086307491
0.4	0.4239860367	0.4239862788	0.4239862886	0.4239862891
0.5	0.5543370146	0.5543399110	0.5543400885	0.5543401061
0.7	0.8992199875	0.8992296944	0.899242462	0.8992475639
0.9	1.4814889672	1.481778951	1.482022918	1.48237693
1.0	1.9416721356	1.959421042	1.960056483	1.962154582

LDM, laplace decomposition method;
NDM, natural decomposition method.

The numerical results in Tab.1 show that the second-order Painleve equation is convergence with four iterations by applying NDM. The given comparison in Tab.1 demonstrates that NDM converges more rapidly than LDM and others (VIM, MADM and VIA-1).

4. IMPORTANT RESULTS OF THE PAINLEVE EQUATION I

The Painlevé equation has a number of important results and applications in mathematics and physics. Some of the key results include the following:

The Painlevé I equation is a completely integrable system, meaning that its solutions can be constructed using certain algebraic and analytic methods. This property has led to a deeper understanding of the behaviour of non-linear systems and their applications in physics and mathematics. The solutions of the Painlevé I equation are known as the Painlevé transcendent, which are a special class of functions with remarkable properties. They have been extensively studied in the context of special functions and have applications in a variety of areas, including random matrix theory, non-linear optics and statistical physics. The Painlevé I equation arises in the study of a wide range of physical systems, including the propagation of non-linear waves in fluids, the dynamics of magnetic vortices in superconductors and the behaviour of certain quantum mechanical systems. Its integrability properties make it a powerful tool for understanding the behaviour of these systems. The Painlevé I equation has connections to a number of other important mathematical and physical systems, including the Korteweg–de Vries equation, the using model in statistical physics and the theory of algebraic curves. Understanding these connections has led to new insights into the behaviour of the Painlevé I equation and its applications in various fields.

Overall, the Painlevé I equation represents a significant contribution to the study of integrable systems and their applications in physics and mathematics. Its important results and connections to other systems continue to make it a subject of active research and interest in the scientific community.

5. CONCLUSION

This article applies the NDM and LDM to find the approximate solutions of the second-order Painleve equation. The properties of the NDM are used to obtain a series solution that converges rapidly to the approximate solution. Also, LDM was applied successfully to solve the Painlevé equation. Furthermore, LDM does not require discretisation of the variable and avoids round-off errors. The numerical result shows that NDM is more attractive and powerful than LDM. In the future, we employ NDM and LDM to solve NLPDEs in science and engineering.

The Painlevé equations are a set of NLODEs that are notable for their integrability properties and their appearance in various areas of physics and mathematics. They were first introduced by the French mathematician Paul Painlevé in the early 20th century.

The novelty of this equation is described as follows:

- One of the key novelties of the Painlevé I equation is its integrability properties. Like the other Painlevé equations, it is integrable, meaning that its solutions can be constructed using certain algebraic and analytic methods. This property has led to a deeper understanding of the behaviour of non-linear systems and their applications in physics and mathematics.
- Another important feature of the Painlevé I equation is its appearance in the study of random matrix theory, where it plays a crucial role in the description of the distribution of eigenvalues of certain random matrices. In particular, it is used to describe the scaling limit of the Tracy–Widom distribution, which arises in the study of the largest eigenvalue of random matrices.

Overall, the Painlevé I equation represents a significant contribution to the study of non-linear systems and their applications in mathematics and physics. Its simplicity and wide-ranging connections continue to make it a subject of active research and interest in the scientific community.

Overall, future work on the Painlevé I equation is likely to involve a combination of theoretical and computational approaches, with the goal of understanding its behaviour and applications in a wide range of contexts.

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