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POINTWISE COMPLETENESS AND POINTWISE DEGENERACY OF DESCRIPTOR LINEAR DISCRETE-TIME SYSTEMS WITH DIFFERENT FRACTIONAL ORDERS

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Abstract: The solution to the system of equations of the descriptor linear discrete-time with different fractional orders is derived by the use of the Drazin inverse of matrices. This solution is applied to analysis of the pointwise completeness and the pointwise degeneracy of the descriptor discrete-time linear systems with different fractional orders. Necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of the descriptor discrete –time linear systems with different fractional orders are established. The proposed methods are illustrated by numerical examples.

Key words: drazin inverse, descriptor, discrete-time, linear, system, pointwise completeness, pointwise degeneracy, fractional, different orders

1. INTRODUCTION

Descriptor (singular) linear systems have been considered in [3,5,7,15,19]. The fundamentals of fractional calculus have been given in [22, 23, 13]. The linear systems with fractional orders have been analyzed in [4, 6, 9, 10] and with different fractional orders in [1, 12, 15, 23, 24]. The analysis of differential algebraic equations and its numerical solutions have been analyzed in [20] and the numerical and symbolic computations of generalized inverses in [29]. The T-Jordan canonical form and the T-Drazin inverse based on the T-product have been addressed in [23]. In [21] The multilinear time-invariant descriptor systems have been analyzed in [21]. The descriptor and standard positive linear systems by the use of Drazin inverse has been addressed in [2, 8, 15]. The pointwise degeneracy of autonomous control systems have been considered in [20] and of linear delay-differential systems with nonnilpotent passive matrices in [16]. The pointwise completeness and degeneracy of fractional descriptor discrete-time linear systems by the use of the Drazin inverse matrices have been addressed in [9, 11, 12] and of fractional different orders in [14, 15, 26]. Analysis of the differential-algebraic equations has been analyzed in [19] and the numerical and symbolic computations of the generalized inverses in [27]. The T-Jordan canonical form and T-Drazin inverse based on the T-Jordan canonical form and T-Drazin inverse based on the T-product has been investigated in [21, 22]. The numerical and symbolic computation of the generalized inverses have been analyzed in [27].

In this paper the pointwise completeness and the pointwise degeneracy of descriptor linear discrete-time systems with different orders will be analyzed.

The paper is organized as follows. In Section 2 the Drazin inverse of matrices is applied to find the solution to descriptor linear discrete-time systems with different fractional orders. Necessary and sufficient conditions for the pointwise completeness of the systems with fractional orders are established in Section 3 and the pointwise degeneracy of the systems in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, Z_+- the set of nonnegative integers, I_n- the $n\times n$ identity matrix, $imgP$ – the image of the matrix P.

2. SOLUTION OF THE STATE EQUATIONS OF FRACTIONAL DESCRIPTOR DISCRETE-TIME LINEAR SYSTEMS

Consider the descriptor fractional discrete-time linear system with two different fractional orders

$$E\begin{bmatrix}\Delta^\alpha x_1(i+1)\\ \Delta^\beta x_2(i+1)\end{bmatrix} = A\begin{bmatrix}x_1(i)\\ x_2(i)\end{bmatrix} \text{ and}$$

$$E=\begin{bmatrix}E_1 & 0\\ 0 & E_2\end{bmatrix}, A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \tag{1}$$

where

$0<\alpha,\beta<2$, $i\in Z_+=\{0,1,2,\dots\}$, $x_1(i)\in\Re^{n_1}$ and

$x_2(i)\in\Re^{n_2}$

are the state vectors and

$E_k, A_{kj}\in\Re^{n_k\times n_j}$; $k,j=1,2$.

The fractional difference of α (β) order is defined by [11, 13]

$$\Delta^\alpha x(i)=\sum_{j=0}^{i} c_\alpha(j)x(i-j),$$

$$c_\alpha(j)=(-1)^j\binom{\alpha}{j}=(-1)^j\frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}, c_\alpha(0)=1, j=1,2,\dots \tag{2}$$

In descriptor systems it is assumed that $det\,E = 0$ and the pencil is regular, i.e.

$$det\left[\begin{bmatrix} E_1 z_1 & 0 \\ 0 & E_2 z_2 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right] \neq 0 \text{ for some } z_1, z_2 \in C \quad (3)$$

where C is the field of complex numbers.

Premultiplying (1) by the matrix

$[E\ \mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - A]^{-1}$

we obtain

$$\bar{E}\begin{bmatrix} \Delta^\alpha x_1(i+1) \\ \Delta^\beta x_2(i+1) \end{bmatrix} = \bar{A}\begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix}, i \in Z_+ \quad (4)$$

where

$$\bar{E} = [E\ \mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - A]^{-1}E = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} \\ \bar{E}_{21} & \bar{E}_{22} \end{bmatrix},$$

$$\bar{A} = [E\ \mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - A]^{-1}A = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}. \quad (5)$$

The equation (1) and (4) have the same solution

$$x(i) = \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix}.$$

Lemma 1. If there exist $c_1, c_2 \in C$ such that

$$\bar{E}[\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2)]\bar{E} = \bar{E}^2[\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2)] \quad (6)$$

then

$$\bar{E}\bar{A} = \bar{A}\bar{E}. \quad (7)$$

Proof. From (5) we have

$$\bar{E}\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - \bar{A} = [E\ \mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - A]^{-1} \times [E\ \mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - A]^{-1}A = I_n \quad (8)$$

and

$$\bar{A} = \bar{E}\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - I_n. \quad (9)$$

Using (9) we obtain

$$\bar{E}\bar{A} = \bar{E}\{\bar{E}\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - I_n\} = \bar{E}^2\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - \bar{E} \quad (10)$$

and

$$\bar{A}\bar{E} = \{\bar{E}\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2) - I_n\}\bar{E} = \bar{E}\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2)\bar{E} - \bar{E}. \quad (11)$$

Therefore, if the condition (6) is satisfied then the equation (7) holds.

Remark 1. If $c_1 = c_2 \in C$ then the equality (6) is always satisfied

$$\bar{E}[\mathrm{diag}(I_{n_1}c_1, I_{n_2}c_2)] = \bar{E}c = c\bar{E}. \quad (12)$$

Lemma 2. If the condition (7) is satisfied then

$$\bar{E}\bar{A}^D = \bar{A}^D\bar{E}, \quad (13)$$

$$\bar{E}^D\bar{A} = \bar{A}\bar{E}^D, \quad (14)$$

$$\bar{E}^D\bar{A}^D = \bar{A}^D\bar{E}^D. \quad (15)$$

Proof is given in [13].

Remark 2. If $det\,A \neq 0$ and we assume $c_1 = c_2 = 0$ then

$$\bar{E} = [-A]^{-1}E, \bar{A} = -I_n \quad (16)$$

in this case the condition (7) is satisfied.

Substituting (2) into (4) we obtain

$$\bar{E}\begin{bmatrix} \Delta^\alpha x_1(i+1) \\ \Delta^\beta x_2(i+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{1\alpha} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{2\beta} \end{bmatrix}\begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix} + \sum_{j=2}^{i+1}\begin{bmatrix} I_{n_1}c_\alpha(j+1) & 0 \\ 0 & I_{n_2}c_\beta(j+1) \end{bmatrix}\begin{bmatrix} x_1(i-j+1) \\ x_2(i-j+1) \end{bmatrix}, \quad (17)$$

where $\bar{A}_{1\alpha} = \bar{A}_{11} + \alpha I_{n_1}, \bar{A}_{2\beta} = \bar{A}_{22} + \beta I_{n_2}$.

In particular case when $\bar{E} = I_n$ we have the following theorem.

Theorem 1. The fractional discrete-time linear system (4) with $\bar{E} = I_n$ and initial conditions

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

has the solution

$$x(i) = \Phi_i\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \quad (18)$$

where

$$\Phi_i = \begin{cases} I_n \text{ for } i = 0 \\ \bar{A}\Phi_{i-1} - D_1\Phi_{i-2} - \ldots - D_{i-1}\Phi_0 \text{ for } i = 1,2,\ldots \end{cases} \quad (19a)$$

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, D_k = \begin{bmatrix} I_{n_1}c_\alpha(k+1) & 0 \\ 0 & I_{n_2}c_\beta(k+1) \end{bmatrix}, k = 1,2,\ldots \quad (19b)$$

Proof is given in [13].

If $\bar{E} \neq I_n$ then the Drazin inverse of matrix $\bar{E}$ will be applied to find the solution to the equation (4).

Definition 1. A matrix $\bar{E}^D$ is called the Drazin inverse of $\bar{E} \in \Re^{n \times n}$ if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \quad (20a)$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \quad (20b)$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \quad (20c)$$

where q is the smallest nonnegative integer (called index of $\bar{E}$), satisfying the condition $\mathrm{rank}\,\bar{E}^q = \mathrm{rank}\,\bar{E}^{q+1}$.

The Drazin inverse $\bar{E}^D$ of a square matrix $\bar{E}$ always exists and is unique. If $\bar{det}\,E \neq 0$ then $\bar{E}^D = \bar{E}^{-1}$. The Drazin inverse matrix $\bar{E}^D$ can be computer by the one of known methods [2, 3, 13].

Theorem 2. The descriptor fractional discrete-time linear system (4) with initial conditions $x(0) = \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix} \in Im(\bar{E}^D\bar{E}) = \bar{E}^D\bar{E}v$, $x \in \Re^n$ has the solution

$$x(i) = \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix} = \hat{\Phi}_i\bar{E}^D\bar{E}v, \quad (21)$$

where

$$\hat{\Phi}_i = \begin{cases} I_n \text{ for } i = 0 \\ \hat{A}\hat{\Phi}_{i-1} - \hat{D}_1\hat{\Phi}_{i-2} - \ldots - \hat{D}_{i-1}\hat{\Phi}_0 \text{ for } i = 1,2,\ldots \end{cases} \quad (22a)$$

$$\hat{A} = \bar{E}^D\bar{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix},$$

$$\widehat{D}_k = \bar{E}^D \begin{bmatrix} I_{n_1} c_\alpha(k+1) & 0 \\ 0 & I_{n_2} c_\beta(k+1) \end{bmatrix}, k = 1,2,\ldots \quad (22b)$$

Proof. Taking into account that the equations (1) and (4) have the same solution the proof will be accomplisched by showing that the solution (21) satisfies the equation (4).

Using (21) and (22) we obtain

$$\begin{aligned}&\bar{E}\widehat{\Phi}_{i+1}\bar{E}^D\bar{E}v = \bar{E}\big(\hat{A}\widehat{\Phi}_i - \widehat{D}_1\widehat{\Phi}_{i-1} - \ldots - \widehat{D}_1\widehat{\Phi}_0\big)\bar{E}^D\bar{E}v = \\ &\bar{E}\bar{E}^D\hat{A}(\hat{A}\widehat{\Phi}_{i-1} - \widehat{D}_1\widehat{\Phi}_{i-2} - \ldots - \widehat{D}_{i-1}\widehat{\Phi}_0)\bar{E}^D\bar{E}v \\ &= \hat{A}(\hat{A}\widehat{\Phi}_{i-1} - \widehat{D}_1\widehat{\Phi}_{i-2} - \ldots - \widehat{D}_{i-1}\widehat{\Phi}_0)(\bar{E}^D\bar{E})^2 v = \hat{A}\widehat{\Phi}_i\bar{E}^D\bar{E}v\end{aligned} \quad (23)$$

since (14) and $(\bar{E}^D\bar{E})^2 = \bar{E}^D\bar{E}$. Therefore, the solution of the equation (1) has the form (21).

3. THE POINTWISE COMPLETENESS OF DESCRIPTOR FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS WITH DIFFERENT FRACTIONAL ORDERS

In this section necessary and sufficient conditions for the pointwise completeness of the descriptor discrete-time linear systems with different fractional orders will be established.

Definition 2. The descriptor fractional discrete-time linear system (1) is called pointwise complete at the point $i = q$ if for every final state $x_f \in \Re^n$, there exists an boundary condition $x(0) \in Im\bar{E}\,\bar{E}^D$ such that

$$x(q) = x_f \in Im\bar{E}\,\bar{E}^D. \quad (24)$$

Theorem 3. The descriptor fractional discrete-time linear system (1) is pointwise complete for $i = q$ and every $x_f \in \Re^n \in Im\bar{E}\,\bar{E}^D$ if and only if

$$det\widehat{\Phi}_q \neq 0 \quad (25a)$$

where

$$\widehat{\Phi}_q = \hat{A}\widehat{\Phi}_{q-1} - \widehat{D}_1\widehat{\Phi}_{q-2} - \ldots - \widehat{D}_{q-1}\widehat{\Phi}_0 \quad (25b)$$

and $\hat{A}$, $\widehat{D}_k$ are defined by (22b).

Proof. From (21) for $i = q$ we obtain

$$x_f = x(q) = \widehat{\Phi}_q\bar{E}^D\bar{E}x(0). \quad (26)$$

For given $x_f \in \Re^n \in Im\bar{E}\,\bar{E}^D$ we may find $x(0) \in Im\bar{E}\,\bar{E}^D$ if and only if the condition (25) is satisfed. Therefore, the descriptor fractional system (1) is pointwise complete at the point $i = q$ if and only if the condition (25) is satisfied.

Example 1. Consider the descriptor fractional system (1) for $\alpha = 0.6, \beta = 0.8$ with the matrices

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$n_1 = 1,\ n_2 = 2. \quad (27)$$

We choose $c_1 = c_2 = 1$ and using (5), (27) we obtain

$$\bar{E} = [E\ \mathrm{diag}(c_1, c_2) - A]^{-1}E = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} \\ \bar{E}_{21} & \bar{E}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\bar{A} = [E\ \mathrm{diag}(c_1, c_2) - A]^{-1}A = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (28)$$

The Drazin inverse matrix of $\bar{E}$ has the form

$$\bar{E}^D = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \bar{E}\bar{E}^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (29)$$

In this case

$$\widehat{\Phi}_1 = \hat{A} = \bar{E}^D\bar{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (30)$$

And $x(0) \in Im\bar{E}\,\bar{E}^D = \begin{bmatrix} x_{11}(0) \\ 0 \\ x_{21}(0) \end{bmatrix}$ and $x_{11}(0)$, $x_{21}(0)$ are arbitrary.

Note that the matrix $\widehat{\Phi}_1$ is singular and by Theorem 2 the descriptor fractional system with (27) is not pointwise complete for $q = 1$ and every $x_f \in \Re^3$ of the form $x_f = \begin{bmatrix} x_{11}(t_f) \\ 0 \\ x_{21}(t_f) \end{bmatrix}$ and $x_{11}(t_f)$, $x_{21}(t_f)$ are arbitrary.

Using (25b) for $q = 2$ we obtain

$$\widehat{\Phi}_2 = \hat{A}^2 - \widehat{D}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 - \begin{bmatrix} I_2 c_\alpha(2) & 0 \\ 0 & c_\beta(2) \end{bmatrix} = \begin{bmatrix} 0.12 & 0 & 0 \\ 0 & 0.88 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \quad (31)$$

and

$$\widehat{det\,\Phi}_2 = 8.448 * 10^{-3} \neq 0. \quad (32)$$

Therefore, by Theorem 2 the descriptor fractional system with (27) is pointwise complete for $q = 2$.

4. THE POINTWISE DEGENERACY OF FRACTIONAL DESCRIPTOR LINEAR DISCRETE-TIME SYSTEMS

In this section necessary and sufficient conditions for the pointwise degeneracy of the descriptor discrete-time linear systems with different fractional orders will be established.

Definition 4.1. The descriptor fractional discrete-time linear system (1) is called pointwise degenerated in the direction v for $q = q_f$ if there exists a vector $v \in \Re^n$ such that for all initial conditions $x(0) \in Im\bar{E}\,\bar{E}^D$ the solution of (1) for $q = q_f$ satisfy the condition

$$v^T x_f = 0. \quad (33)$$

Theorem 3. The descriptor fractional continuous-time linear system (1) is pointwise degenerated in the direction $v \in \Re^n$ for $q = q_f$ if and only if

$$det\widehat{\Phi}_q = 0, \quad (34)$$

where $\widehat{\Phi}_q$ is defined by (25b).

Proof. From (4.1) and (26) for $q = q_f$ we have

$$v^T \hat{\Phi}_q x(0) = 0. \quad (35)$$

There exists a nonzero vector $v \in \Re^n$ such that (35) holds for all $x(0) \in Im\, E\, \bar{E}^D$ if and only if the condition (34) is satisfied. Therefore, the descriptor fractional system (1) is pointwise degenerated in the direction $v \in \Re^n$ for $q = q_f$ if the condition (34) is satisfied. □

Remark 2. The vector $v \in \Re^n$ in which the descriptor fractional discrete-time linear system (1) is pointwise degenerated can be computed from the equation

$$v^T \hat{\Phi}_q = 0. \quad (36)$$

Example 2. (Continuation of Example 1) Consider the system (1) for $\alpha = 0.6, \beta = 0.8$ with the matrices (27). In Example 1 it was shown that the matrix $\hat{\Phi}_q$ for $q = 1$ is nonsingular. Therefore, the descriptor fractional system (1) with (27) is pointwise degenerated for $q = 1$ and any direction v.

From (31) and Theorem 3 it follows that the matrix $\hat{\Phi}_2$ is nonsingular. Therefore, by Theorem 3 the system (1) with (27) is not pointwise degenerated for $i = q = 2$.

5. CONCLUDING REMARKS

The Drazin inverse of matrices has been applied to investigation of the pointwise completeness and the pointwise degeneracy of the descriptor linear discrete-time systems with different fractional orders. Necessary and sufficient conditions for the pointwise completeness (Theorem 2) and for the pointwise degeneracy (Theorem 3) of the fractional linear discrete–time systems have been established. The considerations have been illustrated by numerical examples. The presented methods can be extended to the descriptor linear systems with many different fractional orders.

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