

EXPLORING JACOBI ELLIPTIC AND PERIODIC SOLITARY WAVE SOLUTIONS FOR THE FAMILY OF 3-D WBBM EQUATIONS THROUGH THE GENERALIZED APPROACH

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Abstract: In this paper, we analyze the family of three-dimensional Wazwaz-Benjamin-Bona-Mahony (3-D WBBM) equations using the generalized Jacobi elliptic function expansion method. Understanding complicated wave patterns is critical, therefore, the family of 3-D WBBM equations is a valuable mathematical model with application in fluid mechanics, plasma dynamics, biomechanics, and engineering sciences. The main objective of this study is to build several new multiple wave form solutions for the nonlinear three-dimensional Wazwaz-Benjamin-Bona-Mahony (3-D WBBM) equations. By first using the travelling wave transformation, the nonlinear partial differential equation is transformed into an ordinary differential equation. Then the generalized Jacobi elliptic function expansion method is used to execute a finite series expansion of degree n . Due to this analytical method, we obtained numerous new exact solutions, including periodic solutions in Jacobi elliptic function forms and their corresponding Solitary and shock wave solutions in a limit convergence sense when the modulus parameter m approaches 1 and 0. These results are particularly useful for nonlinear science and mathematical physics professionals, since they provide important insights into the structure and evolution of nonlinear waves in a variety of physical circumstances. To improve the physical description of the solutions, several typical wave profiles are offered to provide a comprehensive analysis of the wave characteristics of the solutions in 2-D, 3-D, and contour visualizations were generated using accurate parameters value with the help of Mathematica. Moreover, by generating novel and accurate propagating soliton waveform solutions, the generalized Jacobi elliptic function expansion methods highlight its importance in uncovering key aspects of the model behaviours as well as suggesting potential applications in the study of water waves.

Key words: Soliton wave solutions, family of 3-D WBBM Equations, Jacobi Elliptic Function Expansion technique, Nonlinear Evaluation Equations

1. INTRODUCTION

Nonlinear phenomena occur in various scientific domains, including applied mathematics, physics, and engineering. Nonlinear evolution equations (NLEEs) are being explored in multiple nonlinear areas, such as nonlinear optics, thermal conductivity, fluid mechanics, optical fiber, electromagnetism, quantum theory, and shallow water wave propagation [1–3]. The research on soliton wave solutions of NLEEs is drawing interest. Because the majority of physical systems are nonlinear, researchers have been encouraged to investigate whether exact solutions exist for NLPDEs, specifically nonlinear evolution equations, or evolution equations (EEs).

Numerous numerical and analytical approaches have been developed to address nonlinear partial differential equations. The researchers prefer analytical approaches over numerical methods because they provide an extensive understanding of physical processes and precise insights into system dynamics. However, analytically solving nonlinear partial differential equations (NLPDEs) can be challenging. For this purpose, in recent decades, much development has been done; many reliable and efficient techniques have been presented to achieve the exact NLEE solutions. Like the exp-function approach [4, 5], (G'/G)-expansion technique [6, 7], F-

expansion approach [8, 9], modified extended tanh-function technique [10, 11], the Hirota bilinear technique [12, 13], the extended rational sin-cos and sinh-cosh methods [14], the generalized sine-gordon expansion approach [15], the modified Kudryashov technique [16], Variational Iteration Method [17], Improved (G'/G)-Expansion technique [18, 19], Riccati-Bernoulli Sub ODE approach [20] Generalized Exponential Rational Function approach [21] and so on. These methods have been effectively established and provided to find a precise solution for NLPDEs. Although these approaches have made great progress, but they often fail when used to more complex models, such as a family of 3-D WBBM equations, where more efficient and extended methodologies are needed.

This article uses an efficient and effective strategy to generate different types of periodic and solitary wave solutions for the three-dimensional WBBM equations. The three-dimensional WBBM equation can be stated as follows:

$$v_x + v^2 v_y + v_t - v_{xzt} = 0, \quad (1.1)$$

$$v_y + v^2 v_x + v_t - v_{xyt} = 0, \quad (1.2)$$

$$v_z + v^2 v_z + v_t - v_{xxt} = 0. \quad (1.3)$$

Wazwaz proposed the 3-D Wazwaz-Benjamin-Bona-Mahony (3-D WBBM) equation in 2017 [22]. The 3-D WBBM equation is the

modified form of the famous KdV equation, which is used to study the stability and soliton-like properties of fluid waves in various scientific phenomena. The family of Wazwaz- Benjamin-Bona-Mahony (3-D WBBM) equations has attracted much attention because of their mathematical structure and practical applications. This equation is characterized by a scenario that addresses nonlinearity and dispersion, making it valuable for analyzing wave phenomena in a physical setting. In recent years, several successful techniques have been applied to study the WBBM model, such as the first integral method [23], the modified Exp-function approach [24], the Generalized rational exponential function approach for analytical solutions [25], Rational sine-Gordon expansion approach [26], the Sardar-Sub equation approach [27], the Improved Bernoulli Sub-Equation Function technique [28], the extended modified auxiliary equation mapping (EMAEM) approach [29], a new extended direct algebraic technique [30], the Φ_6 expansion approach and modified extended direct algebraic [31], the Khater methods [32], and the expanded tanh approach [33]. In addition to the existing analytical methods, the generalized Jacobi elliptic function expansion method, previously applied to various nonlinear PDEs and FPDEs [34–42], has not yet been employed for the 3D WBBM model, offering potential for innovative exact solutions.

In this paper, we integrate the generalized Jacobi elliptic function expansion technique for studying the family of 3-D WBBM equations. The generalized JEFE method is more efficient and generalized than the existing methods for solving the 3-D WBBM equations. The generalized JEFE approach makes good use of Jacobi elliptic functions, allowing for a large number of solutions, such as Jacobian elliptic, trigonometric, and hyperbolic trigonometric functions type of solutions, which improves its application to diverse nonlinear wave equations. Furthermore, it is a noteworthy invention that constitutes notable progress. When solving nonlinear partial differential equations (PDEs), the generalized Jacobi elliptic function expansion technique has several advantages and disadvantages. Its ability to provide periodic and exact solutions is one of its primary features. In order to understand complicated nonlinear dynamics, this capacity is crucial. Equations exhibiting specific symmetries and invariants benefit greatly from this method, as it allows for a more thorough examination of the solutions. Its limited application, however, is one of its main drawbacks. The exact solutions demonstrated that our suggested WBBM equations are perfectly suitable for producing innovative traveling wave structures in a wide range of physical scenarios without any problems, in addition, the article contains details 2-D, contour, and 3-D graphical visualizations of the solutions to demonstrate how well the suggested method solves difficult nonlinear equations and to help readers better understand their physical features, to providing useful information for further research in mathematical physics and shallow water waves.

The following is the order of the work in this paper. Section 2 explains how to solve NLPDEs using generalized JEFE techniques. In Section 3, we describe how this approach can be used to solve the nonlinear 3-D WBBM system. This section is divided into three subsections, each of which presents Periodic wave solutions in terms of Jacobi elliptic function expansion, and solitary and shock wave solutions. Section 4 presents a graphical presentation and discussion. Section 5 presents the conclusion of the study.

2. SUMMARY OF THE GENERALIZED JEFE APPROACH

Within this part of the article, we explain the details of the generalized JEFE approach. The following steps will be followed when

conducting these types of studies:

Step A: Nonlinear partial differential equations (NPDEs) typically take the mathematical expression

$$N\left(v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial z}, \frac{\partial^2 v}{\partial x \partial x}, \frac{\partial^2 v}{\partial x \partial z}, \frac{\partial^2 v}{\partial z \partial z}, \dots\right) = 0. \quad (2.1)$$

Step B: Reformulating Equation (2.1) and utilizing the chain rule, we consider:

$$v(x, y, z, t) = v(\psi),$$

$$\text{where } \psi = px + qy + rz - st, \quad (2.2)$$

where p, q, r , and s represent constants. We can use equation (2.2) to convert the nonlinear PDEs (2.1) into ordinary differential equations.

$$N\left(v, \frac{\partial v}{\partial \psi}, \frac{\partial^2 v}{\partial \psi^2}, \dots\right) = 0. \quad (2.3)$$

The primary objective of this extended indirect strategy is to maximize the possibility of addressing the derived ordinary differential equation, more especially the initial kind of Jacobian problem that involves three variables, such as m_2, m_1 , and m_0 . The goal is to come up with a more comprehensive collection of Jacobi elliptic solutions for the problem that has been presented. It is possible to visually represent the auxiliary equation as follows:

$$(N')^2(\psi) = m_2 N^4(\psi) + m_1 N^2(\psi) + m_0, \quad (2.4)$$

where

$$N' = \frac{dN}{d\psi}, \psi = \psi(x, y, z, t), \text{ and } m_2, m_1, \text{ and } m_0$$

are constants.

The mathematical expressions for Jacobi elliptic functions are given by:

$$sn(\psi) = sn(\psi, k), cn(\psi) = cn(\psi, k), dn(\psi) = dn(\psi, k)$$

Here, the k is used to signify the modulus, and $k \in (0, 1)$.

The solutions of equation (2.4) are presented in Tab.1.

Tab.1. The following table summarizes all the potential solutions to equation (2.3) for the specific m_2, m_1 , and m_0 values that have been provided

NO.	m_2	m_1	m_0	N
1	k^2	$-(1+k^2)$	1	sn
2	k^2	$-(1+k^2)$	1	cd
3	$-k^2$	$2k^2-1$	$1-k^2$	cn
4	-1	$2-k^2$	k^2-1	dn
5	1	$-(1+k^2)$	k^2	ns
6	1	$-(1+k^2)$	k^2	dc
7	$1-k^2$	$2k^2-1$	$-k^2$	nc
8	k^2-1	$2-k^2$	-1	nd
9	$1-k^2$	$2-k^2$	1	sc
10	$-k^2(1-k^2)$	$2k^2-1$	1	sd
11	1	$2-k^2$	$1-k^2$	cs
12	1	$2k^2-1$	$-k^2(1-k^2)$	ds
13	$\frac{-1}{4}$	$\frac{k^2+1}{2}$	$\frac{-(1-k^2)^2}{4}$	$kcn \mp dn$
14	$\frac{1}{4}$	$\frac{-2k^2+1}{2}$	$\frac{1}{4}$	$ns \mp cs$
15	$\frac{1-k^2}{4}$	$\frac{k^2+1}{2}$	$\frac{1-k^2}{4}$	$nc \mp sc$
16	$\frac{1}{4}$	$\frac{k^2-2}{2}$	$\frac{k^4}{4}$	$ns \mp ds$
17	$\frac{k^2}{4}$	$\frac{k^2-2}{2}$	$\frac{k^2}{4}$	$sn \mp icn$
18	$\frac{k^2}{4}$	$\frac{k^2-2}{2}$	$\frac{k^2}{4}$	$\frac{sn}{\sqrt{1-k^2sn \mp cn}}$

19	$\frac{k^2}{4}$	$\frac{k^2-2}{2}$	$\frac{k^2}{4}$	$kcn \pm idn$
20	$\frac{1}{4}$	$\frac{1-2k^2}{2}$	$\frac{1}{4}$	$\frac{sn}{1 \mp cn}$
21	$\frac{k^2}{4}$	$\frac{k^2-2}{2}$	$\frac{1}{4}$	$\frac{sn}{1 \mp dn}$
22	$\frac{k^2-1}{4}$	$\frac{k^2+1}{2}$	$\frac{k^2-1}{4}$	$\frac{dn}{1 \mp ksn}$
23	$\frac{1-k^2}{4}$	$\frac{k^2+1}{2}$	$\frac{-k^2+1}{4}$	$\frac{cn}{1 \mp sn}$
24	$\frac{(1-k^2)^2}{4}$	$\frac{k^2+1}{2}$	$\frac{1}{4}$	$\frac{sn}{dn \mp cn}$
25	$\frac{k^4}{4}$	$\frac{k^2-2}{2}$	$\frac{1}{4}$	$\frac{cn}{\sqrt{1-k^2} \mp dn}$

Tab.2. The Jacobi elliptic function exhibit specific limiting behaviour as $k \rightarrow 0$ and $k \rightarrow 1$

No.	Function	$k \rightarrow 1$	$k \rightarrow 0$
1	$sn(v)$	$\tanh(v)$	$\sin(v)$
2	$cn(v)$	$\operatorname{sech}(v)$	$\cos(v)$
3	$dn(v)$	$\operatorname{sech}(v)$	1
4	$cd(v)$	1	$\cos(v)$
5	$sd(v)$	$\sinh(v)$	$\sin(v)$
6	$nd(v)$	$\cosh(v)$	1
7	$dc(v)$	1	$\sec(v)$
8	$nc(v)$	$\cosh(v)$	$\sec(v)$
9	$sc(v)$	$\sinh(v)$	$\tan(v)$
10	$ns(v)$	$\coth(v)$	$\csc(v)$
11	$ds(v)$	$\operatorname{csch}(v)$	$\csc(v)$
12	$cs(v)$	$\operatorname{csch}(v)$	$\cot(v)$

As $k \rightarrow 0$ and $k \rightarrow 1$, Jacobi elliptic functions, as mentioned in Table. simplify into periodic, trigonometric, and hyperbolic functions. Consequently, soliton solutions and solutions involving trigonometric functions are obtained for the problem. Using the application of the generalized Jacobi elliptic function expansion generalized JEFE technique, $v(\psi)$ is formulated as a finite series of Jacobi elliptic functions

$$v(\psi) = \sum_{i=0}^n a_i N^i(\psi), i = 0, 1, 2, \dots n. \quad (2.5)$$

Where $N(\psi)$ provides the solution to the nonlinear ODE (2.4) with a_i being constants that will be determined subsequently.

The integer n can be derived from equation (2.5) by considering the maximum order derivative component:

$$O\left(\frac{d^\alpha v}{d\psi^\alpha}\right) = n + \alpha, \alpha = 0, 1, 2, 3, \dots \quad (2.6)$$

and the nonlinear component with the highest power of the differential equation

$$O\left(\frac{v^\beta d^\alpha v}{d\psi^\alpha}\right) = (\beta + 1)n + \alpha, \beta = 1, 2, 3, \quad (2.7)$$

in equation (2.3).

Using equation (2.5) and setting all the coefficients corresponding to powers of N equal to 0, we establish a series of nonlinear algebraic equations corresponding to the coefficient a_i of equation (2.5) Utilising Maple, we resolve this system and substitute all the parameters for m_2, m_1 from equation (2.4) into Table 1. This approach integrates the value of equation (2.5) with the selected auxiliary equation, allowing for the derivation of accurate solutions for equation (2.1).

3. MATHEMATICAL FORMULATION OF THE GENEERIALIZED JEFE APPROACH

3.1. Analysis of the first WBBM equation

Let the first three-dimensional WBBM equation be expressed as:

$$\partial_t u + \partial_x u + \partial_y u^3 - \partial_{xzt}^3 u = 0, \quad (3.1)$$

where ∂ represents partial derivative. Utilizing the wave transformation:

$$v(x, y, z, t) = v(\psi), \quad (3.2)$$

where

$$\psi = px + qy + rz - st,$$

and substituting into Eq. (3.2) we obtain:

$$(-c + p)v' + q(v^3)' + prcv''' = 0. \quad (3.3)$$

Integrating Equation (3.3) w.r.t ψ yields:

$$(-c + p)v + qv^3 + prcv'' + C = 0. \quad (3.4)$$

Here, the C acts as the arbitrary constant arising from integration. For simplicity, setting $C = 0$, we have:

$$(-c + p)v + qv^3 + prcv'' = 0. \quad (3.5)$$

Considering the homogeneous balancing of the derivative component of maximum order v'' and the nonlinear component v^3 in Eq. (3.5), it is clear that $n = 1$. Thus, the method suggests the following supplementary solution:

$$v(\psi) = b_0 + b_1 N(\psi). \quad (3.6)$$

Differentiating Eq. (3.6) with respect to ψ , we have:

$$v'(\psi) = b_1' N'(\psi). \quad (3.7)$$

Substituting this into the governing equation, we get:

$$\begin{aligned} v'(\psi) &= b_1 (m_2 N^4(\psi) + m_1 N^2(\psi) + m_0)^{\frac{1}{2}} \\ v''(\psi) &= b_1 N(\psi) (2m_2 N^2(\psi) + m_1). \end{aligned} \quad (3.8)$$

Substitute Eq. (3.8) and Eq. (3.7) into the equation

$$\begin{aligned} -s b_0 - s b_1 N(\psi) + p b_0 + p b_1 N(\psi) + q b_0^3 + 3 q b_0^2 b_1 N(\psi) + 3 q b_0 b_1^2 N(\psi)^2 + q b_1^3 N(\psi)^3 + prs b_1 N(\psi) m_0 + 2 prs b_1 N(\psi) m_2 = 0. \end{aligned} \quad (3.9)$$

By collecting various powers of $N^i(\psi)$, a subsequent set of algebraic equations is derived:

$$-s b_0 + p b_0 + q b_0^3 = 0, \quad (3.10)$$

$$-s b_1 + p b_1 + 3 q b_0^2 b_1 + p r s b_1 m_1 = 0, \quad (3.11)$$

$$3 q b_0 b_1^2 = 0, \quad (3.12)$$

$$q b_1^3 + 2 p r s b_1 m_2 = 0. \quad (3.13)$$

The system described above of algebraic equations is calculated through MAPLE, and we derive the roots of the coefficients involved in equation (3.6):

$$b_0 = 0, \quad (3.14)$$

$$b_1 = \pm \sqrt{\frac{2r m_2}{-q + q p r m_1}} p, \quad (3.15)$$

$$s = \frac{p}{1 - pr m_1}. \quad (3.16)$$

The following solutions for the 3-D WBBM equation can be obtained by inserting the corresponding values into Eq. (3.6):

$$v(x, y, z, t) = \pm \sqrt{\frac{2r m_2}{-q + q p r m_1}} p N \left(px + qy + rz - \frac{pt}{1 - pr m_1} t \right). \quad (3.17)$$

3.2. Analytical periodic solutions in terms of Jacobi elliptic functions (JEF)

Using the data provided in Table 1 and Table 2 and combining the corresponding values as per Eq. (3.6), we may derive the Jacobi elliptic function solutions which are of a periodic nature for Eq. (1.1) as shown below.

$$v_{1,1}(x, y, z, t) = \pm \sqrt{\frac{-2k^2 r}{(pr(k^2 + 1) + q)}} p sn \left(px + qy + rz - \frac{pt}{1 + pr(k^2 + 1)} \right), \quad (3.18)$$

$$v_{1,2}(x, y, z, t) = \pm \sqrt{\frac{-2k^2 r}{(pr(k^2 + 1) + q)}} p cd \left(px + qy + rz - \frac{pt}{1 + pr(k^2 + 1)} \right), \quad (3.19)$$

$$v_{1,3}(x, y, z, t) = \pm \sqrt{\frac{-2k^2 r}{(pr(2k^2 - 1) + q)}} p cn \left(px + qy + rz - \frac{pt}{pr(2k^2 - 1) + 1} \right), \quad (3.20)$$

$$v_{1,4}(x, y, z, t) = \pm \sqrt{\frac{2r}{(pr(1 - k^2) + q)}} p dn \left(px + qy + rz - \frac{pt}{pr(2 - k^2) + 1} \right), \quad (3.21)$$

$$v_{1,5}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(q - pr(k^2 + 1))}} p ns \left(px + qy + rz - \frac{pt}{1 - pr(k^2 + 1)} \right), \quad (3.22)$$

$$v_{1,6}(x, y, z, t) = \pm \sqrt{\frac{-2r}{q(pr(k^2 + 1) + 1)}} p dc \left(px + qy + rz - \frac{pt}{pr(-k^2 - 1) + 1} \right), \quad (3.23)$$

$$v_{1,7}(x, y, z, t) = \pm \sqrt{\frac{-2r(k^2 - 1)}{q(pr(2k^2 - 1) - 1)}} p nc \left(px + qy + rz - \frac{pt}{pr(2k^2 - 1) + 1} \right), \quad (3.24)$$

$$v_{1,8}(x, y, z, t) = \pm \sqrt{\frac{-2r(k^2 - 1)}{(prq(2k^2 - 2) + q)}} p nd \left(px + qy + rz - \frac{pt}{pr(2 - k^2) + 1} \right), \quad (3.25)$$

$$v_{1,9}(x, y, z, t) = \pm \sqrt{\frac{2r(k^2 - 1)}{(prq(k^2 - 2) + q)}} p sc \left(px + qy + rz - \frac{pt}{pr(2 - k^2) + 1} \right), \quad (3.26)$$

$$v_{1,10}(x, y, z, t) = \pm \sqrt{\frac{2k^2 r(k^2 - 1)}{(prq(2k^2 - 1) - q)}} p sd \left(px + qy + rz - \frac{pt}{pr(2k^2 - 1) + 1} \right), \quad (3.27)$$

$$v_{1,11}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(q(pr(k^2 - 2) + 1))}} p cs \left(px + qy + rz - \frac{pt}{pr(2 - k^2) + 1} \right), \quad (3.28)$$

$$v_{1,12}(x, y, z, t) = \pm \sqrt{\frac{2r}{(prq(2k^2 - 1) - q)}} p ds \left(px + qy + rz - \frac{pt}{pr(2k^2 - 1) + 1} \right), \quad (3.29)$$

$$v_{1,13}(x, y, z, t) = \pm \sqrt{\frac{-r}{(q(pr(k^2 + 1) - 2))}} p \left[kcn \left(px + qy + rz - \frac{pt}{pr \frac{(k^2 + 1)}{2} + 1} \right) \pm dn \left(px + qy + rz - \frac{pt}{pr \frac{(k^2 + 1)}{2} + 1} \right) \right], \quad (3.30)$$

$$v_{1,14}(x, y, z, t) = \pm \sqrt{\frac{-r}{(q(pr(1 - 2k^2) - 2))}} p \left[ns \left(px + qy + rz - \frac{pt}{1 - pr(1 - 2k^2)} \right) \pm cs \left(px + qy + rz - \frac{pt}{1 - pr(1 - 2k^2)} \right) \right], \quad (3.31)$$

$$v_{1,15}(x, y, z, t) = \pm \sqrt{\frac{-r(k^2 - 1)}{(q(pr(k^2 + 1) - 2))}} p \left[nc \left(px + qy + rz - \frac{pt}{1 - pr(k^2 + 1)} \right) \pm sc \left(px + qy + rz - \frac{pt}{1 - pr(k^2 + 1)} \right) \right], \quad (3.32)$$

$$v_{1,16}(x, y, z, t) = \pm \sqrt{\frac{r}{(prq(k^2 - 2) - 2q)}} p \left[ns \left(px + qy + rz - \frac{pt}{1 - pr(k^2 - 2)} \right) \pm ds \left(px + qy + rz - \frac{pt}{1 - pr(k^2 - 2)} \right) \right], \quad (3.33)$$

$$v_{1,17}(x, y, z, t) = \pm \sqrt{\frac{k^2 r}{(q(pr(k^2 - 2) - 2))}} p \left[sn \left(px + qy + rz - \frac{pt}{1 - pr(k^2 - 2)} \right) \pm icn \left(px + qy + rz - \frac{pt}{1 - pr(k^2 - 2)} \right) \right], \quad (3.34)$$

$$v_{1,18}(x, y, z, t) = \pm \sqrt{\frac{k^2 r}{(q(pr(k^2 - 2) - 2))}} p \frac{sn \left(px + qy + rz - \frac{pt}{1 - pr(k^2 - 2)} \right)}{(\sqrt{(1 - k^2)} \pm cn \left(px + qy + rz - \frac{pt}{1 - pr(k^2 - 2)} \right))}, \quad (3.35)$$

$$v_{1,19}(x, y, z, t) = \pm \sqrt{\frac{k^2 r}{(prq(2k^2 - 2) + 2q)}} p \left[kcn \left(px + \right. \right.$$

$$\left(qy + rz - \frac{pt}{\left(1 - \frac{pr(1-2k^2)}{2}\right)} \right) \pm idn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(1-2k^2)}{2}\right)} \right) \right], \quad (3.36)$$

$$v_{1,20}(x, y, z, t) = \pm \sqrt{\frac{-r}{(prq(1 - 2k^2) + 2q)}} \frac{sn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(1-2k^2)}{2}\right)} \right)}{1 \pm cn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(1-2k^2)}{2}\right)} \right)}, \quad (3.37)$$

$$v_{1,21}(x, y, z, t) = \pm \sqrt{\frac{k^2 r}{(pqr(k^2 - 2) - 2q)}} p \frac{sn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2-2)}{2}\right)} \right)}{(\sqrt{(1-k^2)} \pm d n \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2-2)}{2}\right)} \right))}, \quad (3.38)$$

$$v_{1,22}(x, y, z, t) = \pm \sqrt{\frac{r(k^2 - 1)}{q(pr(k^2 + 1) - 2)}} p \frac{dn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2+1)}{2}\right)} \right)}{1 \pm ksn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2+1)}{2}\right)} \right)}, \quad (3.39)$$

$$v_{1,23}(x, y, z, t) = \pm \sqrt{\frac{-(rk^2 - r)}{(pqr(k^2 + 1) - 2q)}} p \frac{cn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2+1)}{2}\right)} \right)}{1 \pm sn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2+1)}{2}\right)} \right)}, \quad (3.40)$$

$$v_{1,24}(x, y, z, t) = \pm \sqrt{\frac{-r(k^2 - 1)^2}{(pqr(k^2 + 1) - 2q)}} p \frac{sn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2+1)}{2}\right)} \right)}{dn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2+1)}{2}\right)} \right) \pm cn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2+1)}{2}\right)} \right)}, \quad (3.41)$$

$$v_{1,25}(x, y, z, t) = \pm \sqrt{\frac{k^4 r}{(q(pr(k^2 - 2) - 2))}} p$$

$$\frac{cn \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2-2)}{2}\right)} \right)}{(\sqrt{(1-k^2)} \pm d n \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pr(k^2-2)}{2}\right)} \right))}. \quad (3.42)$$

3.2.1. Solitary wave solutions

When $k \rightarrow 1$, in this category, see in table 2, the solutions $v_{1,7}, v_{1,8}, v_{1,9}, v_{1,10}, v_{1,15}, v_{1,22}, v_{1,23}$ and $v_{1,24}$ become zero. The remaining solutions represent solitary wave solutions and can be determined as follows:

$$v_{1,26}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(2prq+q)}} p \tanh \left(px + qy + rz - \frac{pt}{2pr+1} \right), \quad (3.43)$$

$$v_{1,27}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(prq-q)}} p \operatorname{sech} \left(px + qy + rz - \frac{pt}{pr+1} \right), \quad (3.44)$$

$$v_{1,28}(x, y, z, t) = \pm \sqrt{\frac{2r}{(-prq+q)}} p \operatorname{sech} \left(px + qy + rz - \frac{pt}{pr+1} \right), \quad (3.45)$$

$$v_{1,29}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(2prq+q)}} p \coth \left(px + qy + rz - \frac{pt}{-2pr+1} \right), \quad (3.46)$$

$$v_{1,30}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(-prq+q)}} p \operatorname{csch} \left(px + qy + rz - \frac{pt}{-pr+1} \right), \quad (3.47)$$

$$v_{1,31}(x, y, z, t) = \pm \sqrt{\frac{r}{(prq-q)}} p \operatorname{csch} \left(px + qy + rz - \frac{pt}{pr+1} \right), \quad (3.48)$$

$$v_{1,32}(x, y, z, t) = \pm \sqrt{\frac{-r}{(2prq-2q)}} p \left[\operatorname{sech} \left(px + qy + rz - \frac{pt}{pr+1} \right) \pm \operatorname{sech} \left(px + qy + rz - \frac{pt}{pr+1} \right) \right], \quad (3.49)$$

$$v_{1,33}(x, y, z, t) = \pm \sqrt{\frac{r}{(prq+2q)}} p \left[\coth \left(px + qy + rz - \frac{pt}{\frac{1}{2}pr+1} \right) \pm \operatorname{csch} \left(px + qy + rz - \frac{pt}{\frac{1}{2}pr+1} \right) \right], \quad (3.50)$$

$$v_{1,34}(x, y, z, t) = \pm \sqrt{\frac{r}{(-prq-2q)}} p \left[\coth \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right) \pm \operatorname{csch} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right) \right], \quad (3.51)$$

$$v_{1,35}(x, y, z, t) = \pm \sqrt{\frac{r}{(-prq-2q)}} p \left[\tanh \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right) \pm \operatorname{sech} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right) \right], \quad (3.52)$$

$$v_{1,36}(x, y, z, t) = \pm \sqrt{\frac{r}{(-prq-2q)}} p \frac{\tanh \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right)}{\operatorname{sech} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right)}, \quad (3.53)$$

$$v_{1,37}(x, y, z, t) = \pm \sqrt{\frac{-r}{(prq+2q)}} p \left[\operatorname{sech} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right) \pm \operatorname{sech} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right) \right], \quad (3.54)$$

$$v_{1,38}(x, y, z, t) = \pm \sqrt{\frac{-r}{(-prq+2q)}} p \frac{\tanh \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right)}{1 \pm \operatorname{sech} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right)}, \quad (3.55)$$

$$v_{1,39}(x, y, z, t) = \pm \sqrt{\frac{-r}{(-prq-2q)}} p \frac{\tanh \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right)}{1 \pm \operatorname{sech} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right)}, \quad (3.56)$$

$$v_{1,40}(x, y, z, t) = \pm \sqrt{\frac{r}{(-prq-2q)}} p \frac{\operatorname{sech} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right)}{\operatorname{sech} \left(px + qy + rz - \frac{pt}{1+\frac{1}{2}pr} \right)}. \quad (3.57)$$

3.2.2. Shock wave solutions

When $k \rightarrow 0$, in this category see in table 2, the solutions $v_{1,2}$, $v_{1,3}$, $v_{1,10}$, $v_{1,17}$, $v_{1,18}$, $v_{1,21}$, and $v_{1,25}$ become zero. The remaining solutions represent solitary wave solutions and can be determined as follows:

$$v_{1,41}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(prq+q)}} p \csc \left(px + qy + rz - \frac{pt}{-pr+1} \right), \quad (3.58)$$

$$v_{1,42}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(prq+q)}} p \sec \left(px + qy + rz - \frac{pt}{-pr+1} \right), \quad (3.59)$$

$$v_{1,43}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(prq+q)}} p \sec \left(px + qy + rz - \frac{pt}{-pr+1} \right), \quad (3.60)$$

$$v_{1,44}(x, y, z, t) = \pm \sqrt{\frac{r}{(2prq-q)}} p \tan \left(px + qy + rz - \frac{pt}{2pr+1} \right), \quad (3.61)$$

$$\frac{pt}{2pr+1}), \quad (3.61)$$

$$v_{1,45}(x, y, z, t) = \pm \sqrt{\frac{2r}{(2prq-q)}} p \cot \left(px + qy + rz - \frac{pt}{2pr+1} \right), \quad (3.62)$$

$$v_{1,46}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(prq+q)}} p \csc \left(px + qy + rz - \frac{pt}{-pr+1} \right), \quad (3.63)$$

$$v_{1,47}(x, y, z, t) = \pm \sqrt{\frac{-r}{(prq-2q)}} p \left[\csc \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right) \pm \cot \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right) \right], \quad (3.64)$$

$$v_{1,48}(x, y, z, t) = \pm \sqrt{\frac{-r}{(prq-2q)}} p \left[\sec \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right) \pm \tan \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right) \right], \quad (3.65)$$

$$v_{1,49}(x, y, z, t) = \pm \sqrt{\frac{-2r}{(prq+2q)}} p \left[\csc \left(px + qy + rz - \frac{pt}{1+pr} \right) \pm \csc \left(px + qy + rz - \frac{pt}{1+pr} \right) \right], \quad (3.66)$$

$$v_{1,50}(x, y, z, t) = \pm \sqrt{\frac{r}{(prq-2q)}} p \frac{\sin \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right)}{1 \pm \cos \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right)}, \quad (3.67)$$

$$v_{1,51}(x, y, z, t) = \pm \sqrt{\frac{r}{(prq-2q)}} p \frac{\cos \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right)}{1 \pm \sin \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right)}, \quad (3.68)$$

$$v_{1,52}(x, y, z, t) = \pm \sqrt{\frac{r}{(prq-2q)}} p \frac{\sin \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right)}{1 \pm \cos \left(px + qy + rz - \frac{pt}{1-\frac{1}{2}pr} \right)}. \quad (3.69)$$

3.3. Analysis of the Second WBBM Equation

Using the traveling wave transformation equation (3.2) into equation (1.2), we convert the nonlinear partial differential equation (PDE) to an ordinary differential equation (ODE) of the following form:

$$(-s + r)v' + pv^3 + pqs v'' = 0. \quad (3.70)$$

By the balancing procedure, we find the value of $n = 1$. Thus, the ansatz solution has the following simplified form:

$$v(\psi) = c_0 + c_1 N(\psi). \quad (3.71)$$

Substituting equation (3.71) combine with equation (2.4) into

(3.70), we get:

$$-s c_0 - s c^1 N(\psi) + p c_0 + p c_1 N(\psi) + q c_0^3 + 3q c_0^2 c_1 N(\psi) + 3q c_0 c_1^2 N(\psi)^2 + q c_1^3 N(\psi)^3 + p r s c_1 N(\psi) m_0 + 2p r s c_1 N(\psi) m_2 = 0. \quad (3.72)$$

By collecting various powers of $N^i(\psi)$, a subsequent system of algebraic equations is derived:

$$-s c_0 + p c_0 + q c_0 = 0, \quad (3.73)$$

$$-s c_1 + p c_1 + 3q c_0 c_1 + p r s c_1 m_1 = 0, \quad (3.74)$$

$$3q c_0 c_1^2 = 0, \quad (3.75)$$

$$q c_1^3 + 2p r s c_1 m_2 = 0. \quad (3.76)$$

The system described above of algebraic equations is calculated through MAPLE, and we derive the roots of the coefficients involved in equation (3.71):

$$c_0 = 0, \quad (3.77)$$

$$c_1 = \pm \sqrt{\frac{2qr m_2}{-1 + p q m_1}}, \quad (3.78)$$

$$s = \frac{r}{1 - p q m_1}. \quad (3.79)$$

The following solutions for the first 3-D WBBM equation can be obtained by inserting the corresponding values into Eq. (3.71):

$$v_{x,y,z,t} = \pm \sqrt{\frac{2qr m_2}{-1 + p q m_1}} p N \left(p x + q y + r z - \sqrt{\frac{2qr m_2}{-1 + p q m_1}} t \right). \quad (3.80)$$

3.3.1. Analytical periodic solutions in terms of Jacobi elliptic functions (JEF)

Using the data provided in Tables 1 and 2, and combining the corresponding values as per Eq.(3.71), we may derive the Jacobi elliptic function solutions which are periodic for equation (1.3) as shown below.

$$v_{2,1}(x, y, z, t) = \pm \sqrt{\frac{2qr k^2}{-1 - p q (k^2 + 1)}} p sn \left(p x + q y + r z - \frac{r}{1 + p q (k^2 + 1)} t \right), \quad (3.81)$$

$$v_{2,2}(x, y, z, t) = \pm \sqrt{\frac{2qr k^2}{-1 - p q (k^2 + 1)}} p cd \left(p x + q y + r z - \frac{r}{1 + p q (k^2 + 1)} t \right), \quad (3.82)$$

$$v_{2,3}(x, y, z, t) = \pm \sqrt{\frac{-2k^2 qr}{(p q (2k^2 - 1) - 1)}} p cn \left(p x + q y + r z - \frac{r}{(1 - p q (2k^2 - 1))} t \right), \quad (3.83)$$

$$v_{2,4}(x, y, z, t) = \pm \sqrt{\frac{-2qr}{(p q (2 - k^2) - 1)}} p dn \left(p x + q y + r z - \frac{r}{(1 - p q (2 - k^2))} t \right), \quad (3.84)$$

$$v_{2,5}(x, y, z, t) = \pm \sqrt{\frac{2qr}{(-p q (k^2 + 1) - 1)}} p ns \left(p x + q y + r z - \frac{r}{(1 + p q (k^2 + 1))} t \right), \quad (3.85)$$

$$v_{2,6}(x, y, z, t) = \pm \sqrt{\frac{2qr}{(-p q (k^2 + 1) - 1)}} p dc \left(p x + q y + r z - \frac{r}{(1 + p q (k^2 + 1))} t \right), \quad (3.86)$$

$$v_{2,7}(x, y, z, t) = \pm \sqrt{\frac{-2qr(k^2 - 1)}{(p q (2k^2 - 1) - 1)}} p nc \left(p x + q y + r z - \frac{r t}{(1 - p q (2k^2 - 1))} \right), \quad (3.87)$$

$$v_{2,8}(x, y, z, t) = \pm \sqrt{\frac{-2qr(k^2 - 1)}{(p q (k^2 - 2) + 1)}} p nd \left(p x + q y + r z - \frac{r}{(1 - p q (2 - k^2))} t \right), \quad (3.88)$$

$$v_{2,9}(x, y, z, t) = \pm \sqrt{\frac{-2qr(k^2 - 1)}{(p q (k^2 - 2) + 1)}} p sc \left(p x + q y + r z - \frac{r}{(1 - p q (2 - k^2))} t \right), \quad (3.89)$$

$$v_{2,10}(x, y, z, t) = \pm \sqrt{\frac{-2qr(k^2 - 1)}{(p q (2k^2 - 1) - 1)}} p sd \left(p x + q y + r z - \frac{r}{(1 - p q (2k^2 - 1))} t \right), \quad (3.90)$$

$$v_{2,11}(x, y, z, t) = \pm \sqrt{\frac{2qr}{(p q (2 - k^2) - 1)}} p cs \left(p x + q y + r z - \frac{r}{(1 - p q (2 - k^2))} t \right), \quad (3.91)$$

$$v_{2,12}(x, y, z, t) = \pm \sqrt{\frac{2qr}{(p q (2k^2 - 1) - 1)}} p ds \left(p x + q y + r z - \frac{r}{(1 - p q (2k^2 - 1))} t \right), \quad (3.92)$$

$$v_{2,13}(x, y, z, t) = \pm \sqrt{\frac{-qr}{(p q (k^2 + 1) - 2)}} p \left[kcn \left(p x + q y + r z - \frac{r}{\left(1 - \frac{p q (k^2 + 1)}{2}\right)} t \right) \pm dn \left(p x + q y + r z - \frac{r}{\left(1 - \frac{p q (k^2 + 1)}{2}\right)} t \right) \right], \quad (3.93)$$

$$v_{2,14}(x, y, z, t) = \pm \sqrt{\frac{qr}{(p q (-2k^2 + 1) - 2)}} p \left[ns \left(p x + q y + r z - \frac{r}{\left(1 - \frac{p q (-2k^2 + 1)}{2}\right)} t \right) \pm cs \left(p x + q y + r z - \frac{r}{\left(1 - \frac{p q (-2k^2 + 1)}{2}\right)} t \right) \right], \quad (3.94)$$

$$v_{2,15}(x, y, z, t) = \pm \sqrt{\frac{(-q r (k^2 - 1))}{(p q (k^2 + 1) - 2)}} p \left[nc \left(p x + q y + r z - \frac{r}{\left(1 - \frac{p q (k^2 + 1)}{2}\right)} t \right) \pm sc \left(p x + q y + r z - \frac{r}{\left(1 - \frac{p q (k^2 + 1)}{2}\right)} t \right) \right], \quad (3.95)$$

$$v_{2,16}(x, y, z, t) = \pm \sqrt{\frac{q r}{(p q (k^2 - 2) - 2)}} p \left[ns \left(p x + q y + r z - \frac{r}{\left(1 - \frac{p q (k^2 - 2)}{2}\right)} t \right) \pm ds \left(p x + q y + r z - \frac{r}{\left(1 - \frac{p q (k^2 - 2)}{2}\right)} t \right) \right]$$

$$\left. \frac{r}{\left(1 - \frac{pq(k^2-2)}{2}\right)} t \right) \Bigg], \quad (3.96)$$

$$v_{2,17}(x, y, z, t) = \pm \sqrt{\frac{k^2 q r}{(p q (k^2 - 2) - 2)}} p \left[\operatorname{sn} \left(px + qy + rz - \frac{r}{\left(1 - \frac{pq(k^2-2)}{2}\right)} t \right) \pm \operatorname{icn} \left(px + qy + rz - \frac{r}{\left(1 - \frac{pq(k^2-2)}{2}\right)} t \right) \right], \quad (3.97)$$

$$v_{2,18}(x, y, z, t) = \pm \sqrt{\frac{k^2 q r}{(p q (k^2 - 2) - 2)}} p \frac{\operatorname{sn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2-2)}{2}\right)} \right)}{(\sqrt{(1-k^2)} \pm \operatorname{cn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2-2)}{2}\right)} \right))}, \quad (3.98)$$

$$v_{2,19}(x, y, z, t) = \pm \sqrt{\frac{-qr}{(pq(2k^2-1)+2)}} p \left[\operatorname{kc} \left(px + qy + rz - \frac{r}{\left(1 - \frac{pq(1-2k^2)}{2}\right)} t \right) \pm \operatorname{idn} \left(px + qy + rz - \frac{r}{\left(1 - \frac{pq(1-2k^2)}{2}\right)} t \right) \right], \quad (3.99)$$

$$v_{2,20}(x, y, z, t) = \pm \sqrt{\frac{-q r}{(pq(2k^2-1)+2)}} p \frac{\operatorname{sn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(1-2k^2)}{2}\right)} \right)}{1 \pm \operatorname{cn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(1-2k^2)}{2}\right)} \right)}, \quad (3.100)$$

$$v_{2,21}(x, y, z, t) = \pm \sqrt{\frac{-k^2 q r}{(pq(k^2-2)-2)}} p \frac{\operatorname{sn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2-2)}{2}\right)} \right)}{1 \pm \operatorname{dn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2-2)}{2}\right)} \right)}, \quad (3.101)$$

$$v_{2,22}(x, y, z, t) = \pm \sqrt{\frac{q r (k^2 - 2)}{(pq(k^2 + 1) - 2)}} p \frac{\operatorname{dn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2+1)}{2}\right)} \right)}{1 \pm \operatorname{ksn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2+1)}{2}\right)} \right)}, \quad (3.102)$$

$$v_{2,23}(x, y, z, t) = \pm \sqrt{\frac{-q r (k^2 - 2)}{(pq(k^2 + 1) - 2)}} p \frac{\operatorname{cn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2+1)}{2}\right)} \right)}{1 \pm \operatorname{sn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2+1)}{2}\right)} \right)}, \quad (3.103)$$

$$v_{2,24}(x, y, z, t) = \pm \sqrt{\frac{q r (k^2 - 2)^2}{(pq(k^2 + 1) - 2)}} p \frac{\operatorname{sn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2+1)}{2}\right)} \right)}{\operatorname{dn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2+1)}{2}\right)} \right) \pm \operatorname{cn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2+1)}{2}\right)} \right)}, \quad (3.104)$$

$$v_{2,25}(x, y, z, t) = \pm \sqrt{\frac{-k^4 q r}{(pq(k^2 - 2) - 2)}} p \frac{\operatorname{cn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2-2)}{2}\right)} \right)}{(\sqrt{(1-k^2)} \pm \operatorname{dn} \left(px + qy + rz - \frac{pt}{\left(1 - \frac{pq(k^2-2)}{2}\right)} \right))}, \quad (3.105)$$

3.3.2. Solitary wave type solutions

When $k \rightarrow 1$, in this category see in table 2, the solution $v_{2,7}, v_{2,8}, v_{2,9}, v_{2,10}, v_{2,15}, v_{2,22}, v_{2,23}$ and $v_{2,24}$ become zero. The remaining solutions represent solitary wave solutions and can be determined as follows:

$$v_{2,26}(x, y, z, t) = \pm \sqrt{\frac{2qr}{-1-2pq}} p \tanh \left(px + qy + rz - \frac{r}{1+2pq} t \right), \quad (3.106)$$

$$v_{2,27}(x, y, z, t) = \pm \sqrt{\frac{-2qr}{pq-1}} p \operatorname{sech} \left(px + qy + rz - \frac{r}{1-pq} t \right), \quad (3.107)$$

$$v_{2,28}(x, y, z, t) = \pm \sqrt{\frac{-2qr}{pq-1}} p \operatorname{sech} \left(px + qy + rz - \frac{r}{1-pq} t \right), \quad (3.108)$$

$$v_{2,29}(x, y, z, t) = \pm \sqrt{\frac{2qr}{-2pq-1}} p \coth \left(px + qy + rz - \frac{r}{1+2pq} t \right), \quad (3.109)$$

$$v_{2,30}(x, y, z, t) = \pm \sqrt{\frac{-2qr}{pq+1}} p \operatorname{csch} \left(px + qy + rz - \frac{r}{1+pq} t \right), \quad (3.110)$$

$$v_{2,31}(x, y, z, t) = \pm \sqrt{\frac{2qr}{-pq-1}} p \operatorname{csch} \left(px + qy + rz - \frac{r}{1+pq} t \right), \quad (3.111)$$

$$v_{2,32}(x, y, z, t) = \pm \sqrt{\frac{-2qr}{2pq-2}} p \left[\operatorname{sech} \left(px + qy + rz - \frac{r}{1-pq} t \right) \pm \operatorname{sech} \left(px + qy + rz - \frac{r}{1-pq} t \right) \right], \quad (3.112)$$

$$v_{2,33}(x, y, z, t) = \pm \sqrt{\frac{qr}{-pq-2}} p \left[\operatorname{coth} \left(px + qy + rz - \frac{r}{1+\frac{1}{2}pq} t \right) \pm \operatorname{csch} \left(px + qy + rz - \frac{r}{1+\frac{1}{2}pq} t \right) \right], \quad (3.113)$$

$$v_{2,34}(x, y, z, t) = \pm \sqrt{\frac{qr}{-pq-2}} p \left[\operatorname{coth} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right) \pm \operatorname{csch} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right) \right], \quad (3.114)$$

$$v_{2,35}(x, y, z, t) = \pm \sqrt{\frac{qr}{pq-2}} p \left[\tanh \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right) \pm \operatorname{isech} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right) \right], \quad (3.115)$$

$$v_{2,36}(x, y, z, t) = \pm \sqrt{\frac{qr}{-pq-2}} p \frac{\tanh \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right)}{\operatorname{sech} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right)}, \quad (3.116)$$

$$v_{2,37}(x, y, z, t) = \pm \sqrt{\frac{-qr}{pq+2}} p \left[\operatorname{sech} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right) \pm \operatorname{isech} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right) \right], \quad (3.117)$$

$$v_{2,38}(x, y, z, t) = \pm \sqrt{\frac{qr}{pq+2}} p \frac{\tanh \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right)}{1 \pm \operatorname{sech} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right)}, \quad (3.118)$$

$$v_{2,39}(x, y, z, t) = \pm \sqrt{\frac{qr}{-pq-2}} p \frac{\tanh \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right)}{1 \pm \operatorname{sech} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right)}, \quad (3.119)$$

$$v_{2,40}(x, y, z, t) = \pm \sqrt{\frac{qr}{-pq-2}} p \frac{\operatorname{sech} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right)}{\operatorname{sech} \left(px + qy + rz + \frac{r}{-1-\frac{1}{2}pq} t \right)}, \quad (3.120)$$

3.3.3. Shock wave solution

When $k \rightarrow 0$, in this category see in table 2, the solutions $v_{2,1}, v_{2,2}, v_{2,3}, v_{2,10}, v_{2,17}, v_{2,18}, v_{2,21}$ and $v_{2,25}$ become zero. The remaining solutions represent solitary wave solutions and can be determined as follows:

$$v_{2,41}(x, y, z, t) = \pm \sqrt{\frac{2qr}{-pq-1}} p \csc \left(px + qy + rz - \frac{r}{1+2pq} t \right), \quad (3.121)$$

$$v_{2,42}(x, y, z, t) = \pm \sqrt{\frac{2qr}{-pq-1}} p \sec \left(px + qy + rz - \frac{r}{1+2pq} t \right), \quad (3.122)$$

$$v_{2,43}(x, y, z, t) = \pm \sqrt{\frac{-2qr(k^2-1)}{-pq-1}} p \operatorname{psec} \left(px + qy + rz - \frac{r}{1+2pq} t \right), \quad (3.123)$$

$$v_{2,44}(x, y, z, t) = \pm \sqrt{\frac{-2qr}{-2pq+1}} p \tan \left(px + qy + rz - \frac{r}{1-2pq} t \right), \quad (3.124)$$

$$v_{2,45}(x, y, z, t) = \pm \sqrt{\frac{-2qr}{-2pq+1}} p \cot \left(px + qy + rz - \frac{r}{1-2pq} t \right), \quad (3.125)$$

$$v_{2,46}(x, y, z, t) = \pm \sqrt{\frac{2qr}{-pq-1}} p \csc \left(px + qy + rz - \frac{r}{1+2pq} t \right), \quad (3.126)$$

$$v_{2,47}(x, y, z, t) = \pm \sqrt{\frac{qr}{pq-2}} p \left[\csc \left(px + qy + rz + \frac{r}{1-\frac{1}{2}pq} t \right) \pm \cot \left(px + qy + rz + \frac{r}{1-\frac{1}{2}pq} t \right) \right], \quad (3.127)$$

$$v_{2,48}(x, y, z, t) = \pm \sqrt{\frac{qr}{pq-2}} p \left[\sec \left(px + qy + rz + \frac{r}{1-\frac{1}{2}pq} t \right) \pm \tan \left(px + qy + rz + \frac{r}{1-\frac{1}{2}pq} t \right) \right], \quad (3.128)$$

$$v_{2,49}(x, y, z, t) = \pm \sqrt{\frac{qr}{-2pq-2}} p \left[\csc \left(px + qy + rz + \frac{r}{1+pq} t \right) \pm \csc \left(px + qy + rz + \frac{r}{1+pq} t \right) \right], \quad (3.129)$$

$$v_{2,50}(x, y, z, t) = \pm \sqrt{\frac{-qr}{-pq+2}} p \frac{\sin \left(px + qy + rz - \frac{r}{1-\frac{1}{2}pq} t \right)}{1 \pm \cos \left(px + qy + rz - \frac{r}{1-\frac{1}{2}pq} t \right)}, \quad (3.130)$$

$$v_{2,51}(x, y, z, t) =$$

$$\pm \sqrt{\frac{qr}{pq-2}} p \frac{\cos\left(px + qy + rz - \frac{r}{1-\frac{1}{2}pq}t\right)}{1 \pm \sin\left(px + qy + rz - \frac{r}{1-\frac{1}{2}pq}t\right)}, \quad (3.131)$$

$$v_{2,52}(x, y, z, t) = \pm \sqrt{\frac{qr}{pq-2}} p \frac{\sin\left(px + qy + rz - \frac{r}{1-\frac{1}{2}pq}t\right)}{\cos\left(px + qy + rz - \frac{r}{1-\frac{1}{2}pq}t\right)}, \quad (3.132)$$

3.4. Analysis of the Third WBBM Equation

Using the traveling wave transformation Equation (3.1) into Equation (1.3), we convert the nonlinear partial differential equation (PDE) to an ordinary differential equation (ODE) of the following form:

$$(-s + r)v' + pv^3 + pqsv'' = 0. \quad (3.133)$$

By the balancing procedure, we find the value of $n = 1$. Thus, the ansatz solution has the following simplified form:

$$v(\psi) = d_0 + d_1 N(\psi). \quad (3.134)$$

Substituting Eq. (2.4) in Eq.(3.134), we get

$$-sd_0 - sd_1 N(\psi) + pd_0 + pd_1 N(\psi) + qd_0^3 + 3qd_0^2 d_1 N(\psi) + 3qd_0 c_1^2 N(\psi)^2 + qd_1^3 N(\psi)^3 + prsd_1 N(\psi)m_0 + 2prsd_1 N(\psi)m_2 = 0. \quad (3.135)$$

By collecting various powers of $N(\psi)$, a subsequent system of algebraic equations is derived:

$$-sd_0 + pd_0 + qd_0^3 = 0, \quad (3.136)$$

$$-sd_1 + pd_1 + 3qd_0^2 d_1 + prsd_1 m_1 = 0, \quad (3.137)$$

$$3qd_0 d_1^2 = 0, \quad (3.138)$$

$$qd_1^3 + 2prsd_1 m_2 = 0. \quad (3.139)$$

The system described above of algebraic equations is calculated through MAPLE, and we derive the roots of the coefficients involved in equation (3.134):

$$d_0 = 0, \quad (3.140)$$

$$d_1 = \pm \sqrt{\frac{qm_2}{-r+rp^2m_1}} p, \quad (3.141)$$

$$s = -\frac{q}{-1+p^2m_1}. \quad (3.142)$$

The following solutions for the first 3-D WBBM equation can be obtained by inserting the corresponding values into Eq. (3.134):

$$v(x, y, z, t) = \pm \sqrt{\frac{qm_2}{-r+rp^2m_1}} p N\left(px + qy + rz + \frac{q}{p^2m_1-1}t\right). \quad (3.143)$$

3.4.1. Analytical periodic solutions in terms of Jacobi elliptic functions (JEF)

Using the data provided in table 1 and 2, and combining the corresponding values as per Eq. (3.6), we may derive the Jacobi elliptic function solutions which are in a periodic nature for Eq. (3.6) as shown below.

$$v_{3,1}(x, y, z, t) = \pm \sqrt{\frac{-k^2q}{(p^2r(k^2+1)+r)}} p \operatorname{sn}\left(px + qy + rz + \frac{q}{-p^2(k^2+1)-1}t\right), \quad (3.144)$$

$$v_{3,2}(x, y, z, t) = \pm \sqrt{\frac{-k^2q}{(p^2r(k^2+1)+r)}} p \operatorname{cd}\left(px + qy + rz + \frac{q}{-p^2(k^2+1)-1}t\right), \quad (3.145)$$

$$v_{3,3}(x, y, z, t) = \pm \sqrt{\frac{-k^2q}{(p^2r(2k^2-1)-r)}} p \operatorname{cn}\left(px + qy + rz + \frac{q}{p^2(2k^2-1)-1}t\right), \quad (3.146)$$

$$v_{3,4}(x, y, z, t) = \pm \sqrt{\frac{-q}{(p^2r(2-k^2)-r)}} p \operatorname{dn}\left(px + qy + rz + \frac{q}{p^2(2-k^2)-1}t\right), \quad (3.147)$$

$$v_{3,5}(x, y, z, t) = \pm \sqrt{\frac{-q}{(p^2r(k^2+1)+r)}} p \operatorname{ns}\left(px + qy + rz + \frac{q}{-p^2(k^2+1)-1}t\right), \quad (3.148)$$

$$v_{3,6}(x, y, z, t) = \pm \sqrt{\frac{-q}{(p^2r(k^2+1)+r)}} p \operatorname{dc}\left(px + qy + rz + \frac{q}{-p^2(k^2+1)-1}t\right), \quad (3.149)$$

$$v_{3,7}(x, y, z, t) = \pm \sqrt{\frac{-q(k^2-1)}{(p^2r(2k^2-1)-r)}} p \operatorname{nc}\left(px + qy + rz + \frac{q}{p^2(2k^2-1)-1}t\right), \quad (3.150)$$

$$v_{3,8}(x, y, z, t) = \pm \sqrt{\frac{-q(k^2-1)}{(p^2r(2k^2-1)+r)}} p \operatorname{nd}\left(px + qy + rz + \frac{qt}{p^2(2-k^2)-1}\right), \quad (3.151)$$

$$v_{3,9}(x, y, z, t) = \pm \sqrt{\frac{q(k^2-1)}{(p^2r(2k^2-1)+r)}} p \operatorname{sc}\left(px + qy + rz + \frac{q}{p^2(2-k^2)-1}t\right), \quad (3.152)$$

$$v_{3,10}(x, y, z, t) = \pm \sqrt{\frac{k^2q(k^2-1)}{(p^2r(2k^2-1)-r)}} p \operatorname{sd}\left(px + qy + rz + \frac{q}{p^2(2k^2-1)-1}t\right), \quad (3.153)$$

$$v_{3,11}(x, y, z, t) = \pm \sqrt{\frac{-q}{(p^2r(k^2-2r)+r)}} p \operatorname{cs}\left(px + qy + rz + \frac{q}{p^2(2-k^2)-1}t\right), \quad (3.154)$$

$$v_{3,12}(x, y, z, t) = \pm \sqrt{\frac{q}{(p^2r(2k^2-1)-r)}} p \operatorname{ds}\left(px + qy + rz + \frac{q}{p^2(2k^2-1)-1}t\right), \quad (3.155)$$

$$v_{3,13}(x, y, z, t) = \pm \sqrt{\frac{-q}{(p^2r(k^2+1)-2r)}} p \left[\operatorname{kc}n\left(px + qy + rz + \frac{q}{p^2(k^2+1)-1}t\right) \pm \operatorname{dn}\left(px + qy + rz + \frac{q}{p^2(k^2+1)-1}t\right) \right], \quad (3.156)$$

$$v_{3,14}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(p^2r(-2k^2+1)-2r)}} p \left[\operatorname{ns}\left(px + \right. \right.$$

$$qy + rz + \frac{q}{\frac{p^2(-2k^2+1)}{2}-1}t \Big) \pm cs \left(px + qy + rz + \frac{q}{\frac{p^2(-2k^2+1)}{2}-1}t \right) \Big], \quad (3.157)$$

$$v_{3,15}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q(1-k^2)}{(p^2r(k^2+1)-2r)}} p \left[nc \left(px + qy + rz + \frac{q}{\frac{p^2(k^2+1)}{2}-1}t \right) \pm sc \left(px + qy + rz + \frac{q}{\frac{p^2(k^2+1)}{2}-1}t \right) \right], \quad (3.158)$$

$$v_{3,16}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(p^2r(k^2-2)-2r)}} p \left[ns \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right) \pm ds \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right) \right], \quad (3.159)$$

$$v_{3,17}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}k^2q}{(p^2r(k^2-2)-2r)}} p \left[sn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right) \pm icn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right) \right], \quad (3.160)$$

$$v_{3,18}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}k^2q}{(p^2r(k^2-2)-2r)}} p \left[\frac{sn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)}{\sqrt{1-k^2} \pm cn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)} \right], \quad (3.161)$$

$$v_{3,19}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(p^2r(1-2k^2)-2r)}} p \left[kcn \left(px + qy + rz + \frac{q}{\frac{p^2(2k^2-1)}{2}-1}t \right) \pm idn \left(px + qy + rz + \frac{q}{\frac{p^2(2k^2-1)}{2}-1}t \right) \right], \quad (3.162)$$

$$v_{3,20}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(p^2r(1-2k^2)-2r)}} p \left[\frac{sn \left(px + qy + rz + \frac{q}{\frac{p^2(2k^2-1)}{2}-1}t \right)}{1 \pm cn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-1)}{2}-1}t \right)} \right], \quad (3.163)$$

$$v_{3,21}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}k^2q}{(p^2r(k^2-2)-2r)}} p$$

$$\left[\frac{sn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)}{1 \pm dn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)} \right], \quad (3.164)$$

$$v_{3,22}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q(k^2-1)}{r(p^2(k^2+1)-2)}} p \left[\frac{dn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)}{1 \pm ksn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)} \right], \quad (3.165)$$

$$v_{3,23}(x, y, z, t) = \pm \sqrt{\frac{q(1-k^2)}{2r(p^2(k^2+1)-1)}} p \left[\frac{cn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)}{1 \pm sn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)} \right], \quad (3.166)$$

$$v_{3,24}(x, y, z, t) = \pm \sqrt{\frac{q(1-k^2)^2}{(p^2r(k^2+1)-2r)}} \cdot \frac{sn \left(px + qy + rz - \frac{qt}{\left(\frac{p^2(k^2+1)}{2}-1 \right)} \right)}{dn \left(px + qy + rz - \frac{qt}{\left(\frac{pq(k^2+1)}{2}-1 \right)} \right) \pm cn \left(px + qy + rz - \frac{qt}{\left(\frac{p^2(k^2+1)}{2}-1 \right)} \right)}, \quad (3.167)$$

$$v_{3,25}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}k^4q}{(p^2r(k^2-2)-2r)}} p \left[\frac{cn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)}{\sqrt{1-k^2} \pm dn \left(px + qy + rz + \frac{q}{\frac{p^2(k^2-2)}{2}-1}t \right)} \right], \quad (3.168)$$

3.4.2 Solitary wave type solutions

When $k \rightarrow 1$, in this category see in table 2, the solution $v_{3,7}, v_{3,8}, v_{3,9}, v_{3,10}, v_{3,15}, v_{3,22}, v_{3,23}$ and $v_{3,24}$ become zero. The remaining solutions represent solitary wave solutions and can be determined as follows:

$$v_{3,26}(x, y, z, t) = \pm \sqrt{\frac{-q}{(2p^2r+r)}} p \tanh \left(px + qy + rz + \frac{q}{-2p^2-1}t \right), \quad (3.169)$$

$$v_{3,27}(x, y, z, t) = \pm \sqrt{\frac{-q}{(2p^2r+r)}} p \operatorname{sech} \left(px + qy + rz + \frac{q}{-2p^2-1}t \right), \quad (3.170)$$

$$v_{3,28}(x, y, z, t) = \pm \sqrt{\frac{-q}{(p^2r-r)}} p \operatorname{sech} \left(px + qy + rz + \frac{q}{p^2-1}t \right), \quad (3.171)$$

$$v_{3,29}(x, y, z, t) = \pm \sqrt{\frac{-q}{2p^2r+r}} p \coth \left(px + qy + rz + \frac{q}{2p^2r+r}t \right)$$

$$\frac{q}{-p^2-1}t), \quad (3.172)$$

$$v_{3,30}(x, y, z, t) = \pm \sqrt{\frac{-q}{(-p^2r+r)}}p \operatorname{csch}\left(px + qy + rz + \frac{q}{p^2-1}t\right), \quad (3.173)$$

$$v_{3,31}(x, y, z, t) = \pm \sqrt{\frac{-q}{(2p^2r-2r)}}p \left[\operatorname{sech}\left(px + qy + rz + \frac{q}{p^2-1}t\right) \pm \operatorname{sech}\left(px + qy + rz + \frac{q}{p^2-1}t\right) \right], \quad (3.174)$$

$$v_{3,32}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(-p^2r-2r)}}p \left[\operatorname{coth}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \pm \operatorname{csch}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \right], \quad (3.175)$$

$$v_{3,33}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(-p^2r-2r)}}p \left[\operatorname{coth}\left(px + qy + rz + \frac{r}{\frac{p^2}{2}-1}t\right) \pm \operatorname{csch}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \right], \quad (3.176)$$

$$v_{3,34}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(-p^2r-2r)}}p \left[\tanh\left(px + qy + rz + \frac{r}{\frac{p^2}{2}-1}t\right) \pm \operatorname{isech}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \right], \quad (3.177)$$

$$v_{3,35}(x, y, z, t) = \pm \sqrt{\frac{q}{(-\frac{1}{2}p^2r-r)}}p \left[\frac{\operatorname{sn}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right)}{\pm \operatorname{cn}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right)} \right], \quad (3.178)$$

$$v_{3,36}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(-p^2r-2r)}}p \left[\operatorname{sech}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \pm \operatorname{isech}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \right], \quad (3.179)$$

$$v_{3,37}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(-p^2r-2r)}}p \left[\frac{\tanh\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right)}{1 \pm \operatorname{sech}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right)} \right], \quad (3.180)$$

$$v_{3,38}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(-p^2r-2r)}}p \left[\frac{\tanh\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right)}{1 \pm \operatorname{sech}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right)} \right], \quad (3.181)$$

$$v_{3,39}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{r(-p^2-2)}}p \left[\frac{\operatorname{sech}\left(px + qy + rz + \frac{2q}{p^2-2}t\right)}{\operatorname{sech}\left(px + qy + rz + \frac{2q}{p^2-2}t\right)} \right]. \quad (3.182)$$

3.4.3. Shock wave solution

When $k \rightarrow 0$, in this category see in table 2, the solutions $v_{3,1}, v_{3,2}, v_{3,3}, v_{3,4}, v_{3,10}, v_{3,17}, v_{3,18}, v_{3,21}$ and $v_{3,25}$ become zero. The remaining solutions represent solitary wave solutions and can be determined as follows:

$$v_{3,40}(x, y, z, t) = \pm \sqrt{\frac{-q}{(p^2r+r)}}p \operatorname{csc}\left(px + qy + rz + \frac{q}{-p^2-1}t\right), \quad (3.183)$$

$$v_{3,41}(x, y, z, t) = \pm \sqrt{\frac{-q}{(p^2r+r)}}p \operatorname{sec}\left(px + qy + rz + \frac{q}{-p^2-1}t\right), \quad (3.184)$$

$$v_{3,42}(x, y, z, t) = \pm \sqrt{\frac{-q}{(-p^2r-r)}}p \operatorname{sec}\left(px + qy + rz + \frac{q}{-p^2-1}t\right), \quad (3.185)$$

$$v_{3,43}(x, y, z, t) = \pm \sqrt{\frac{-q}{(-2p^2r+r)}}p \tan\left(px + qy + rz + \frac{q}{2p^2-1}t\right), \quad (3.186)$$

$$v_{3,44}(x, y, z, t) = \pm \sqrt{\frac{-q}{(-2p^2r+r)}}p \cot\left(px + qy + rz + \frac{q}{2p^2-1}t\right), \quad (3.187)$$

$$v_{3,45}(x, y, z, t) = \pm \sqrt{\frac{q}{(-p^2r-r)}}p \operatorname{csc}\left(px + qy + rz + \frac{q}{-p^2-1}t\right), \quad (3.188)$$

$$v_{3,46}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(p^2r-2r)}}p \left[\operatorname{csc}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \pm \cot\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \right], \quad (3.189)$$

$$v_{3,47}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(p^2r-2r)}}p \left[\operatorname{sec}\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \pm \tan\left(px + qy + rz + \frac{q}{\frac{p^2}{2}-1}t\right) \right], \quad (3.190)$$

$$v_{3,48}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(-2p^2r-2r)}} p \left[\csc \left(px + qy + rz + \frac{q}{-p^2-1}t \right) \pm \csc \left(px + qy + rz + \frac{q}{-p^2-1}t \right) \right], \quad (3.191)$$

$$v_{3,49}(x, y, z, t) = \pm \sqrt{\frac{\frac{1}{2}q}{(p^2r-2r)}} p \left[\frac{\sin \left(px + qy + rz + \frac{q}{-p^2-1}t \right)}{1 \pm \cos \left(px + qy + rz + \frac{q}{-p^2-1}t \right)} \right], \quad (3.192)$$

$$v_{3,50}(x, y, z, t) = \pm \sqrt{\frac{q}{(2p^2r-2r)}} p \left[\frac{\cos \left(px + qy + rz + \frac{q}{-p^2-1}t \right)}{1 \pm \sin \left(px + qy + rz + \frac{q}{-p^2-1}t \right)} \right], \quad (3.193)$$

$$v_{3,51}(x, y, z, t) = \pm \sqrt{\frac{q}{(p^2r-2r)}} p \frac{\sin \left(px + qy + rz - \frac{qt}{\left(\frac{p^2}{2}-2\right)} \right)}{1 \pm \cos \left(px + qy + rz - \frac{qt}{\left(\frac{p^2}{2}-1\right)} \right)}. \quad (3.194)$$

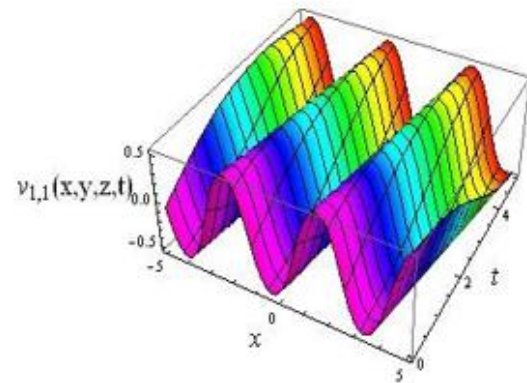
4. GRAPHICAL RESULTS AND DISCUSSION

In this section, we present several solution figures in 2-D, 3-D, and contour plots. All of the figures were created using Mathematica. The generalized JEFE technique yields novel solutions to the nonlinear three-dimensional Wazwaz-Benjamin-Bona-Mahony (3-D WBBM) equations, and its periodicity is proved. We illustrated many sorts of soliton structures, including steep kink, kink, peakon, rogue, and periodic soliton. The 2D plots show simplified cross-sections of the wave solutions, focusing on specific directions to highlight their oscillatory and (quasi-)periodic patterns. These views make it easier to observe changes in amplitude, phase, and frequency over time, helping us understand how the waves behave and evolve.

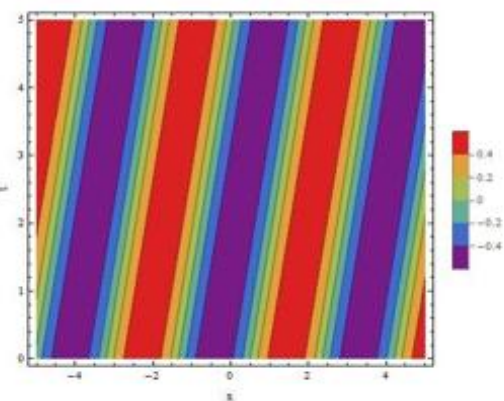
The 3D surface plots clearly display how the wave's height, frequency, and shape vary with different parameters, offering insights that are hard to see just from equations. Contour plots, which map the wave levels onto a flat surface, reveal the internal structure of the solutions, showing areas with steady values or sharp transitions. Together, these 2D, 3D, and contour plots give a fuller picture of the wave behaviors described by the 3D WBBM equations. Subsequently, the generalized JEFE process leads to more consistent results when addressing the WBBM equation, making it an effective tool for generating exact periodic and solitary wave-type soliton solutions.

Additionally, the Φ^6 -expansion and modified extended direct algebraic methods [31], and Khater method [32] to compute trigonometric, hyperbolic, Jacobi elliptic, and rational functions. The expanded tanh approach [33] was also applied to the bright, dark, periodic, and single soliton solutions. To differentiate this from the previous efforts. All our founded solutions are depending on the Jacobi elliptic functions and their corresponding hyperbolic and trigonometric functions. Jacobi elliptic function is a more generalized form of the hyperbolic and trigonometric functions.

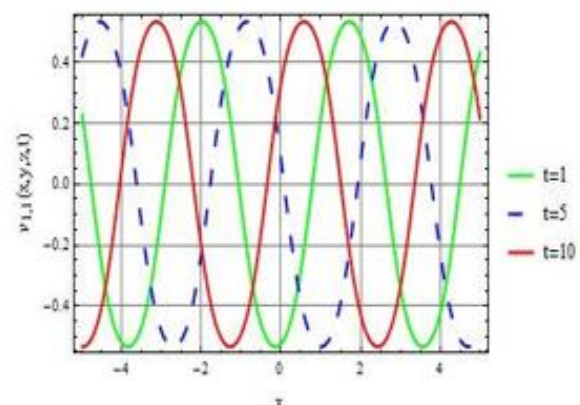
This work is important for our research because it shows how effective the generalized Jacobi elliptic function expansion method is in analyzing and resolving intricate nonlinear systems. The technique advances our knowledge of the dynamics at play in nonlinear systems by producing precise and physically meaningful solutions as well as making complex wave phenomena like solitons easier to visualize. The Jacobi elliptic function expansion approach is a strong and flexible tool for studying nonlinear wave dynamics.



a) 3-D visualization

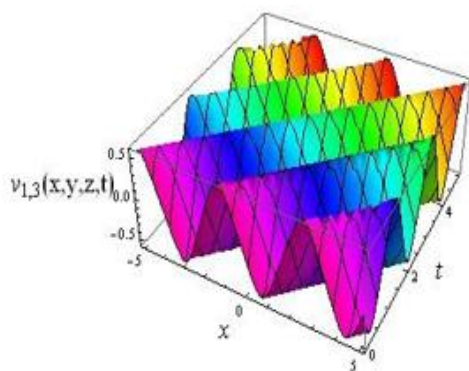


b) Contour figure

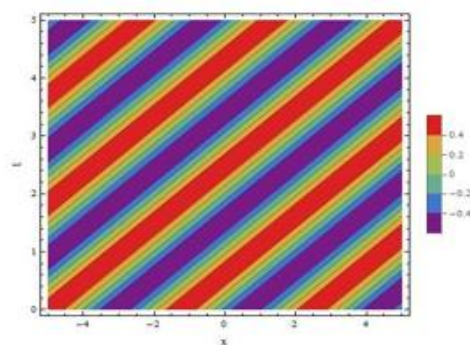


c) The time 2-D plot for t=1,5 and 10

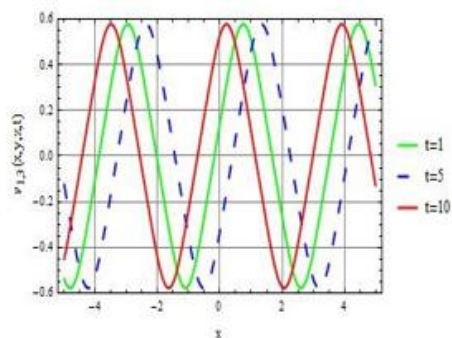
Fig.1. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,1}(x, y, z)$ are presented for the parameter values $k = 0.5, r = 1, p = 2, q = -2$, and $y = z = 1$



a) 3-D visualization

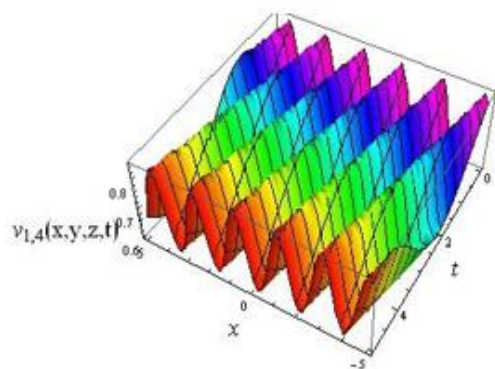


b) contour figure

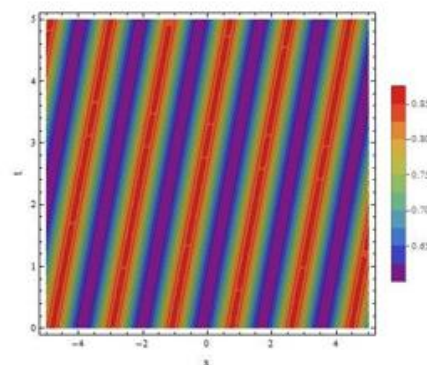


c) the time 2-D plot for $t=1,5$ and 10

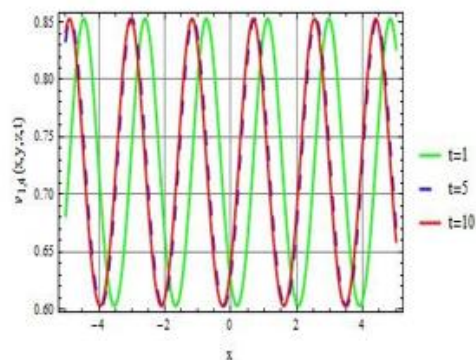
Fig. 2. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,3}(x, y, z)$ are presented for the parameter values $k = 0.5$, $r = 0.5$, $p = 2$, $q = 2$, and $y = z = 1$



a) 3-D visualization

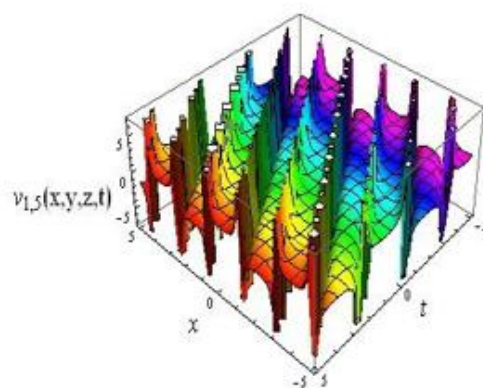


b) contour figure

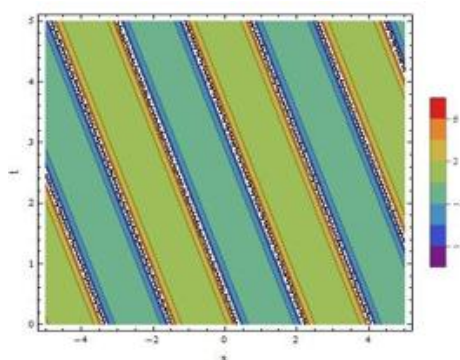


c) the time 2-D plot for $t=1,5$ and 10

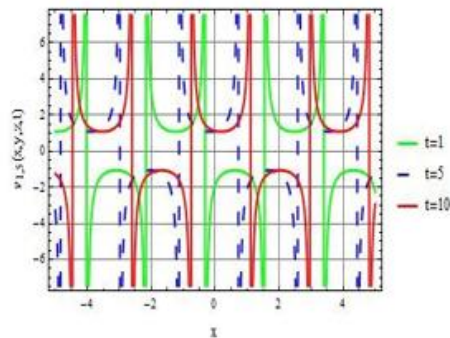
Fig.3. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,4}(x, y, z)$ are presented for the parameter values $k = 0.5$, $r = 0.5$, $p = 2$, $q = -2$, and $y = z = 1$



a) 3-D visualization



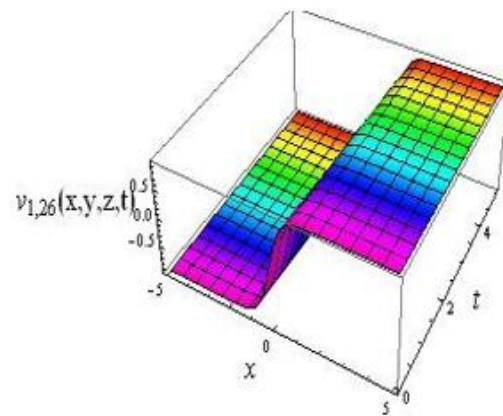
b) contour figure



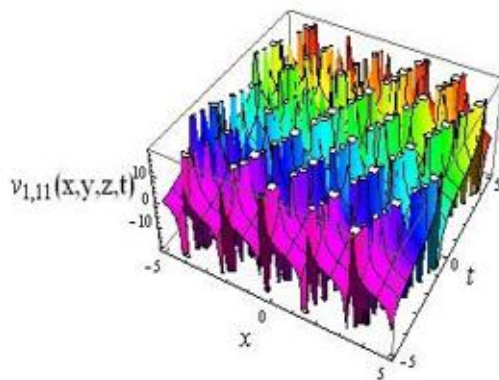
c) the time 2-D plot for $t=1,5$ and 10

Fig.4. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,5}(x,y,z)$ are presented for the parameter values $k = 0.5, r = 1, p = 2, q = 2$, and $y = z = 1$

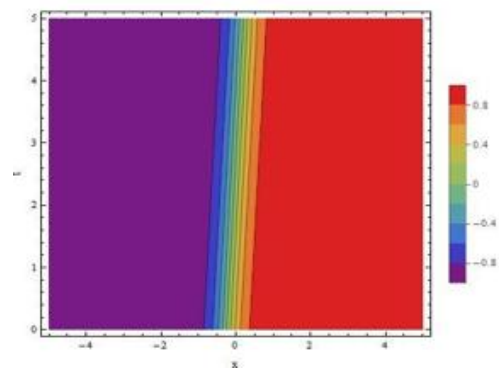
$-2, q = -2$, and $y = z = 1$



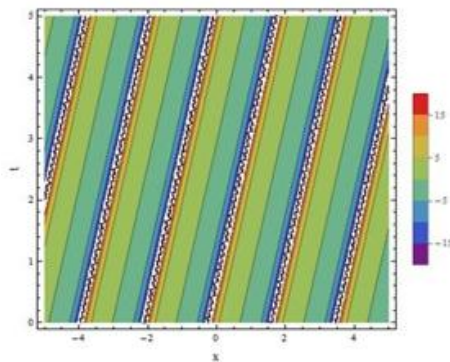
a) 3-D visualization



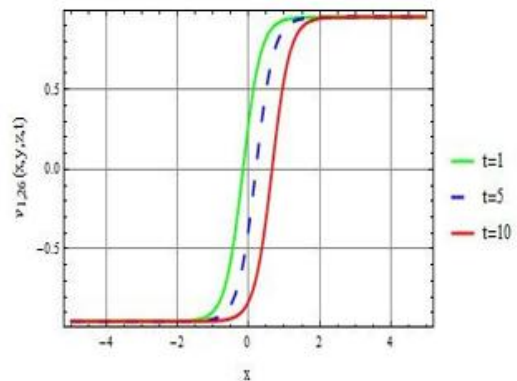
a) 3-D visualization



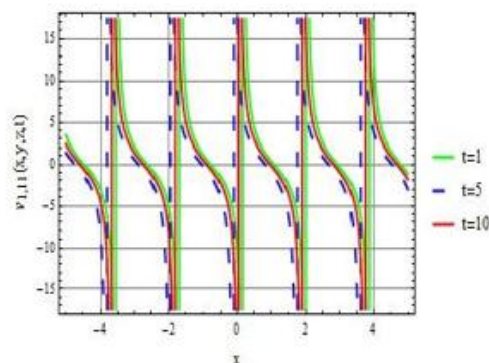
b) contour figure



b) contour figure



c) the time 2-D plot for $t=1,5$ and 10

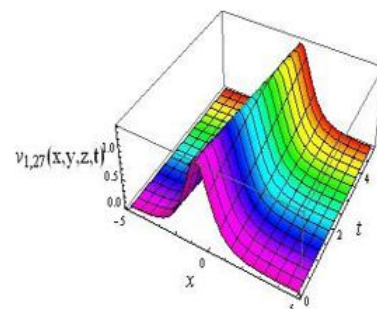


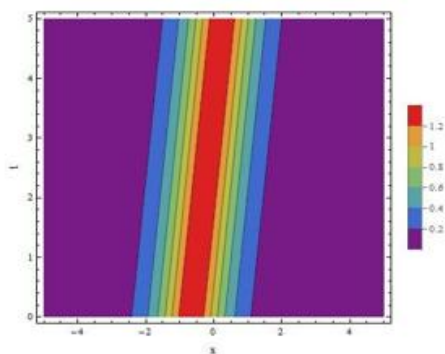
c) the time 2-D plot for $t=1,5$ and 10

Fig.5. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,11}(x,y,z)$ are presented for the parameter values $k = 0.5, r = 1.5, p =$

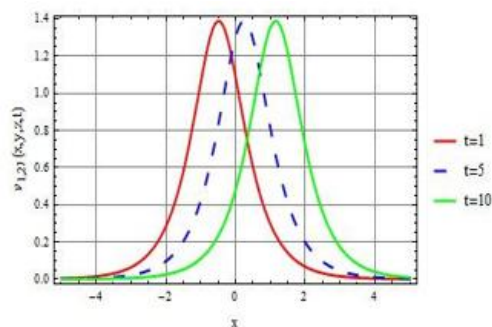
a) 3-D visualization

Fig.6. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,26}(x,y,z)$ are presented for the parameter values $r = 2.5, p = 2, q = -2$, and $y = z = 1$



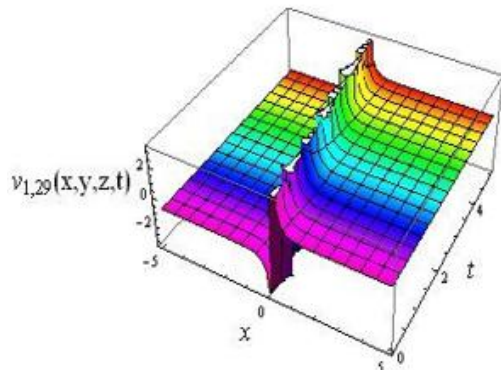


b) contour figure

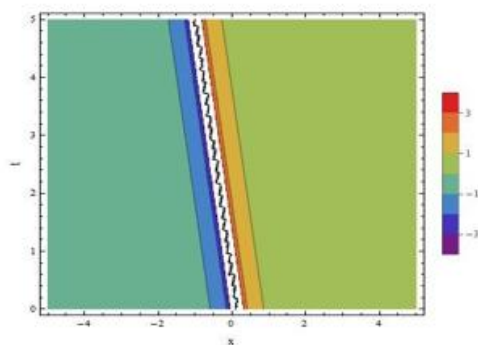


c) the time 2-D plot for $t=1,5$ and 10

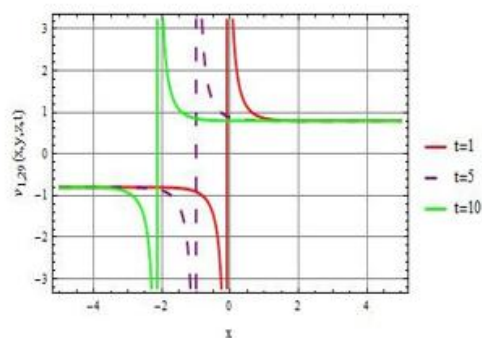
Fig. 7. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,27}(x, y, z)$ are presented for the parameter values $r = 3, p = 1.5, q = -2$, and $y = z = 1$



a) 3-D visualization

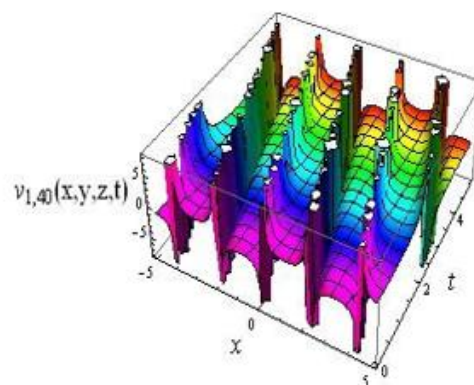


b) contour figure

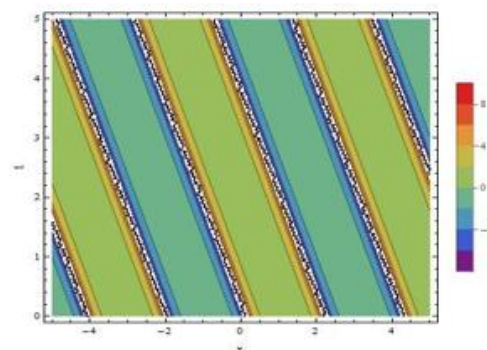


c) the time 2-D plot for $t=1,5$ and 10

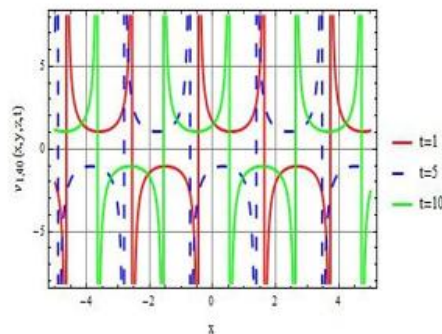
Fig. 8. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,29}(x, y, z)$ are presented for the parameter values $r = 1.8, p = 1.5, q = -2$, and $y = z = 1$



a) 3-D visualization

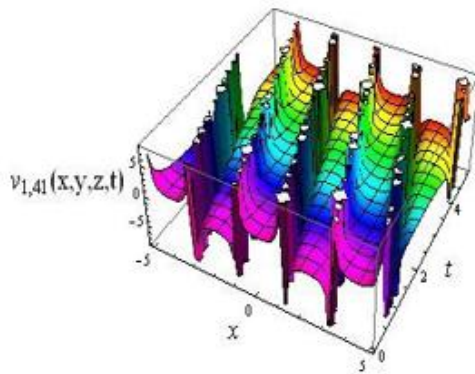


b) contour figure

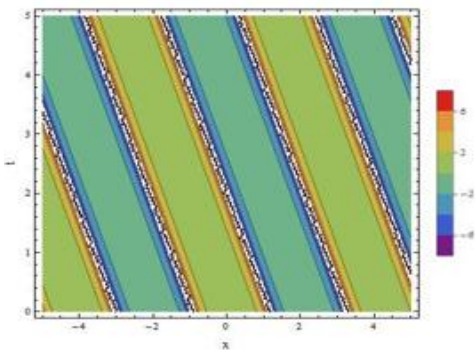


c) the time 2-D plot for $t=1,5$ and 10

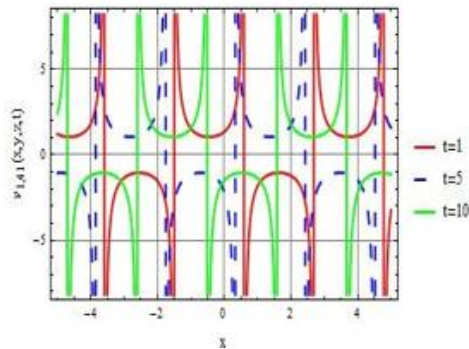
Fig. 9. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,40}(x, y, z)$ are presented for the parameter values $r = 1.8, p = 1.5, q = -2$, and $y = z = 1$



a) 3-D visualization

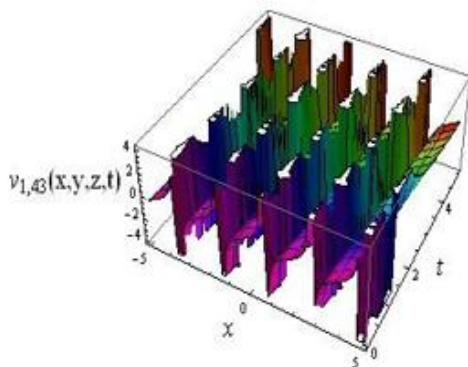


b) contour figure

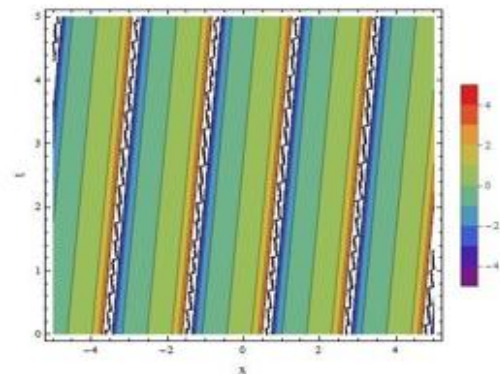


c) the time 2-D plot for t=1,5 and 10

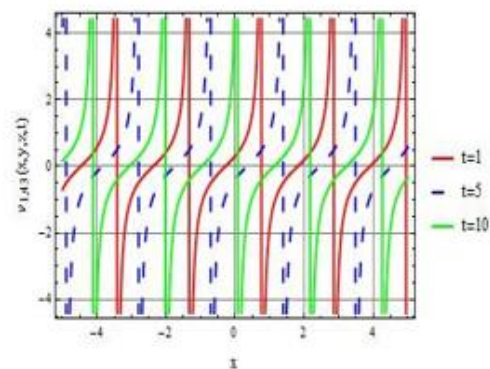
Fig.10. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,41}(x, y, z)$ are presented for the parameter values $r = 1.8$, $p = 1.5$, $q = -2$, and $y = z = 1$



a) 3-D visualization

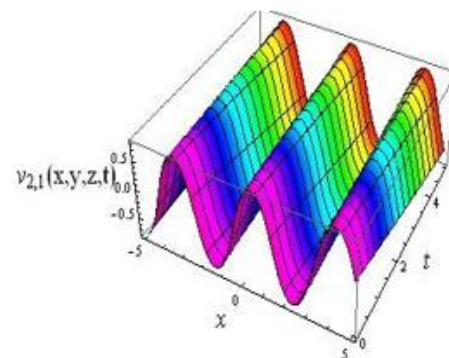


b) contour figure

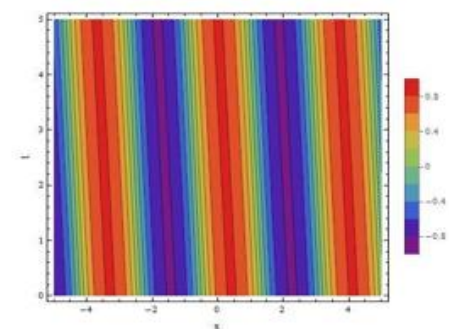


c) the time 2-D plot for t=1,5 and 10

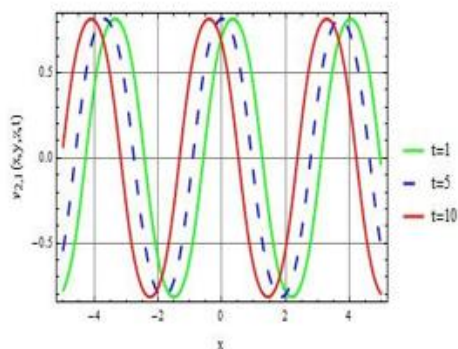
Fig.11. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{1,43}(x, y, z)$ are presented for the parameter values $r = 1.8$, $p = 1.5$, $q = 2$, and $y = z = 1$



a) 3-D visualization

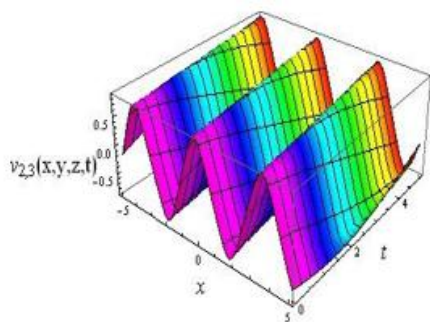


b) contour figure

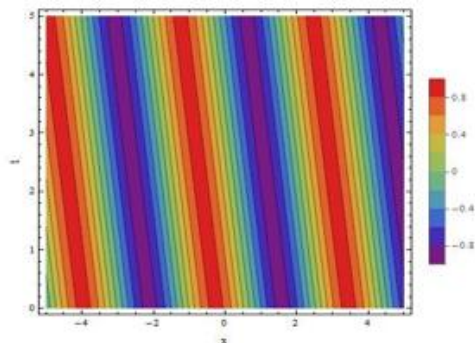


c) the time 2-D plot for $t=1,5$ and 10

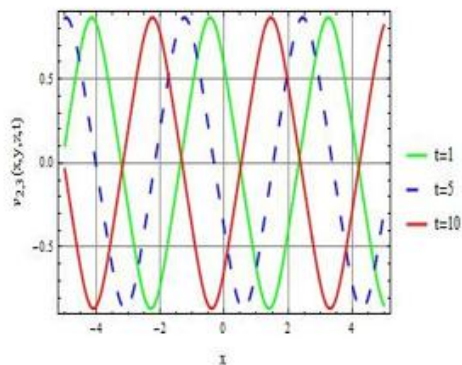
Fig.12. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{2,1}(x, y, z)$ are presented for the parameter values $k = 0.5, r = -1, p = 2, q = 2, y = 1$, and $z = 1$



a) 3-D visualization

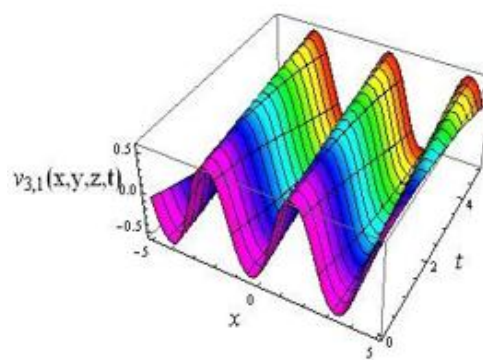


b) contour figure

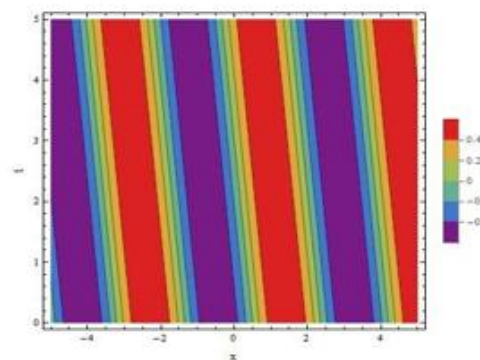


c) the time 2-D plot for $t=1,5$ and 10

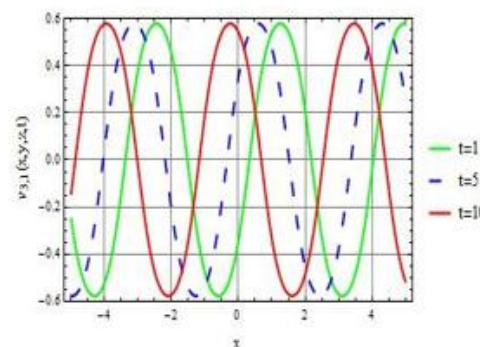
Fig.13. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{2,3}(x, y, z)$ are presented for the parameter values $k = 0.5, r = -1, p = 2, q = 1.5, y = 1$, and $z = 1$



a) 3-D visualization

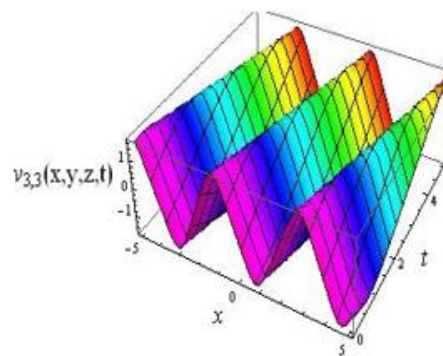


b) contour figure

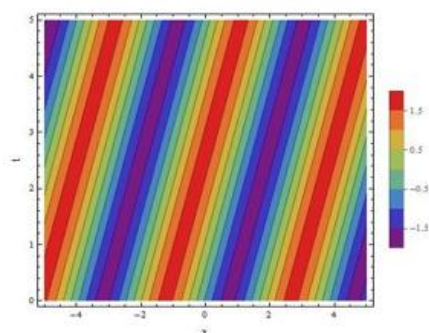


c) the time 2-D plot for $t=1,5$ and 10

Fig.14. The 3-D visualization, contour plot, and 2-D graphical representation of $v_{3,1}(x, y, z)$ are presented for the parameter values $k = 0.5, r = 1, p = 2, q = -2, y = 1$, and $z = 1$



a) 3-D visualization



b) contour figure

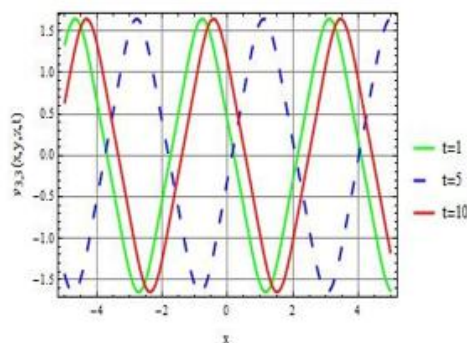

c) the time 2-D plot for $t=1, 5$ and 10

Fig.15. The 3-D visualization, contour plot and 2-D graphical representation of $v_{3,3}(x, y, z)$ are presented for the parameter values $k = 0.6, r = 0.5, p = 2, q = 2$, and $y = z = 1$

5. CONCLUSION

The family of 3-D WBBM equations is efficiently solved in this work through the application of the generalized Jacobi elliptic function expansion technique. Through this method, the new periodic solutions are determined, which incorporates both solitary wave and shock wave solutions that have not been documented previously in the literature. These results give a greater understanding of the rich dynamic phenomena regulated by the family of 3-D WBBM equations, improving their applicability in fluid dynamics, nonlinear optics, plasma physics, and engineering. The obtained results have not been found in previous literature using this approach. To improve the physical description of the solutions several typical wave profiles are offered to provide a comprehensive analysis of the wave characteristics in 2-D, 3-D, and contour visualizations were generated using accurate parameters value with the help of Mathematica. Such graphical visualizations aid in our ability to more clearly understand and comprehend the system's inherent features. The objective of this approach is quite applicative and powerful to analyze various soliton solutions, therefore it may be additionally applicable to many other nonlinear evaluation equations. Future research direction should focus to apply these approaches to additional complicated nonlinear systems to determine their broader application. Furthermore, investigating the stability and interactions of the generated solutions under different initial conditions and parameter changes might provide additional information. Furthermore, incorporating physics-informed neural networks represents a promising avenue for validating these solutions in practical applications, with the potential to bridge the gap between theoretical mathematics and real-world implementations in

fields such as fluid dynamics, plasma physics, and engineering systems.

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