

AN OBSERVATION ON THE ELZAKI TRANSFORM AND THE FRACTIONAL COUPLED SYSTEM OF PDEs

Mohammed E. A. RABIE*, Tarig M. ELZAKI**

*Mathematics Department, Faculty of Sciences and Humanities - Afif, Shaqra University, Saudi Arabia
 Department of Mathematic, Faculty of Sciences, Sudan University of Sciences and Technology, Sudan

**Mathematics Department, Faculty of Sciences, University of Jeddah, Saudi Arabia
 Department of Mathematic, Faculty of Sciences, Sudan University of Sciences and Technology, Sudan

mrabie@su.edu.sa, tarig.alzaki@gmail.com

received 04 February 2025, revised 18 May 2025, accepted 16 June 2025

Abstract: This paper presents a new approach for solving several fractional coupled systems of nonlinear partial differential equations (FCSNLPDEs) using initial conditions (ICs). This approach is based on the Elzaki transform (ET). A comprehensive description is provided to facilitate understanding of the procedure. The applicability and validity of this technique for solving FCSNLPDE problems in a few steps have been demonstrated. Using this approach, both linear and nonlinear FCSPDEs can be solved without the need for discretization or restrictive assumptions. This method requires fewer numerical calculations because it does not introduce approximation errors. Numerical examples are presented to illustrate the accuracy and efficiency of this new technique. To further illustrate how the suggested approach affected the outcomes, 2D and 3D graphs and tables were employed.

Key words: Fractional Coupled System of Nonlinear Partial Differential Equations, Novel Analytical Method, Elzaki Transform, Mittag-Leffler

1. INTRODUCTION

Many applications in physics are simulated using fractional nonlinear PDEs (FNLPEs). Even though a lot of academics are willing to study these models, they insist on using different methods to identify answers that are explicit, exact, or approximate. In recent times, a lot of scholars have been interested in solving ODEs, PDEs, and integral equations. Many authors have also concentrated on investigating various methods for solving NLPDEs.

Although fractional derivatives (FDs) are not new to mathematics, their application in research has lagged for a considerable amount of time. The prevalence of non-equivalent formulation of a fractional derivative may be one reason for its disfavor [1]. Furthermore, because FDs are non-local, they cannot be accurately interpreted geometrically [2]. FDs are capable of explaining a wide range of events, such as the nonlinear oscillation of earthquakes [3] as well as the mistake in the fluid-dynamic traffic model resulting from the belief of continuous traffic flow [4]. According to empirical data, fractional order phenomena in DEs and FPDEs for seepage flow in porous media are proposed by [5] and [6], respectively. Mainardi [7] provides an outline of a few uses of FDs in continuous mechanics and statistical mechanics. Numerous authors have studied the analytical findings on the uniqueness and existence of FDE solutions [1, 8]. In recent times, a variety of methods, including Homotopy Analysis and Adomian decomposition, have been employed to solve dynamic systems that comprise FDs, FDEs, FPDEs, and FIDEs [9–13]. Due to their non-local character, fractional operators (FO) are a valuable tool for analyzing phenomena involving the memory effect. We underline that a PDE can be converted from a local to a nonlocal one by substituting a specific FO for the standard derivative with respect to time.

Some of the excellent studies that have been recently covered is the one conducted by Shabir Ahmad et al. [14–18] are Dynamical study of a novel 4D hyperchaotic system [14], Dynamical Analysis of Bio-Ethanol Production Model under Generalized Nonlocal Operator in Caputo Sense [15], A hybrid analytical technique for solving nonlinear fractional order PDEs [16], Mathematical Analysis of Biodegradation Model under Nonlocal Operator in Caputo Sense [17] and Analysis of the seventh-order Caputo fractional KdV equation [18].

This essay will solve SFPDEs using the innovative ET method. Boundary value problems are resolved with ET and its variations. The novel process gives better accuracy since different initial approximations are used in all iteration of the solution, in contrast to the recommended approach [17], which puts the result in a finite series form which is simple to calculate. A number of the difficulties are addressed using the general explanation of the suggested remedy. Finding analytical solutions with ICs for FCSNLPDEs is challenging. The current work uses a relatively simple to comprehend and implement approach to produce closed-form analytical solutions for the FCSNLPDEs.

The suggested method offers a fresh approach to broadening the use of fractal techniques to a variety of systems with highly accurate results. This method may show great promise in the near future for creating and using intricate fractal models in a variety of domains, including engineering and physics. The freshly developed method has two distinct features: it is simple to use and offers recommendations for choosing the initial iteration, selecting the first iteration is crucial since it can initiate the method's convergence and enable the calculation of the exact solution in a limited number of steps, if the ICs are exactly zero, we can choose the first iteration all or a subset of the inhomogeneous terms.

Because this method relies entirely on the ET transform and initial conditions, it will be ineffective if the ET is missing for the fractional derivative or if the initial conditions are not present. Also, if the initial iteration is not chosen correctly, we may not reach a convergence to the exact solution.

In this paper, we employ a particular analytical method that utilizes the ET to solve FCSNLPDEs. Examples are given to illustrate this method's efficacy and dependability. This method can be used to solve functional equations that arise from numerical modeling of many processes.

Definition 1: Following are the definitions for the Riemann-Liouville (R-L) operator of order

$\kappa > 0$, of $\Phi \in C_\mu, \mu \geq -1$,

$$J^\kappa \Phi(B) = \frac{1}{\Gamma(\kappa)} \int_0^B (B-o)^{\kappa-1} \Phi(o) do, \kappa > 0,$$

$$J^0 \Phi(B) = \Phi(B).$$

Properties of J^α ,

For $\Phi^n \in C_\mu, n \in N, \kappa, \lambda \geq 0$ and $\sigma \geq -1$:

$$J^\kappa J^\lambda \Phi(B) = J^{\kappa+\lambda} \Phi(B),$$

$$J^\kappa B^\sigma = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\kappa+1)} B^{\kappa+\sigma}.$$

Definition 2: As stated by Caputo, the FD of $\Phi(B)$ is:

$$D^\kappa \Phi(B) = J^{m-\kappa} D^m \Phi(B),$$

for $m-1 < \kappa \leq m, m \in N, B > 0$, and $\Phi \in C_{-1}^m$.

To find an FD's correct order, one computes an ordinary derivative first, followed by a fractional integral.

Just like the R-L fractional integral operator, the integer-order integration is a linear process:

$$J^\kappa (\sum_{i=1}^n c_i \Phi_i(B)) = \sum_{i=1}^n c_i J^\kappa \Phi_i(B),$$

$\{c_i\}_{i=1}^n$ are constants.

[10], explains that FD is believed to possess a Caputo meaning in the current inquiry, which supports the application of the Caputo definition. One of the great advantages of the Caputo fractional derivative is that it allows traditional initial and boundary conditions to be included in the formulation of the problem. In addition, its derivative for a constant is zero, therefore, Caputo fractional derivative was chosen in this paper.

2. ELZAKI TRANSFORM

In general, the ET is described as:

$$E[\Phi(\tau)] = o \int_0^\infty \Phi(\tau) e^{-\frac{\tau}{o}} d\tau = T(o), \tau > 0, \quad (1)$$

o is a complex value.

PDEs, ODEs, and integral equations in [1-4] and [23-25, 27-32] can all be resolved with ET. Effective application of ET is possible while Sumudu and Laplace transforms fail to handle DEs having variable coefficients [26].

Theorem 1: [2] The partial derivatives are converted by ET in the following ways:

$$E\left[\frac{\partial \Phi(v, \tau)}{\partial \tau}\right] = \frac{1}{o} T(v, o) - o \Phi(v, 0), E\left[\frac{\partial \Phi(v, \tau)}{\partial v}\right] =$$

$$\frac{d}{dv} [T(v, o)],$$

$$E\left[\frac{\partial^2 \Phi(v, \tau)}{\partial \tau^2}\right] = \frac{1}{o^2} T(v, o) - \Phi(v, 0) - o \frac{\partial \Phi(v, 0)}{\partial \tau}, E\left[\frac{\partial^2 \Phi(v, \tau)}{\partial v^2}\right] = \frac{d^2}{dv^2} [T(v, o)].$$

ET of some functions:

$\Phi(\tau)$	$E[\Phi(\tau)] = T(o)$
1	o^2
τ	o^3
τ^n	$n! o^{n+2}$
$e^{a\tau}$	$\frac{o^2}{1-ao}$
$\sin a \tau$	$\frac{ao^3}{1+a^2o^2}$
$\cos a \tau$	$\frac{o^2}{1+a^2o^2}$

In this instance, we offer a few lemmas that enable function $\Phi(A)$ inference from its ET.

Lemma 1: ET of R-L operator of order $\kappa > 0$ is

$$E[J^\kappa \Phi(A)] = o^\kappa T(o).$$

Proof: We establish by

$$E[J^\kappa \Phi(A)] = E\left[\frac{1}{\Gamma(\kappa)} \int_0^A (A-B)^{\kappa-1} \Phi(B) dB\right] = \frac{1}{\Gamma(\kappa)} \frac{1}{o} T(o) G(o) = o^\kappa T(o)$$

where

$$G(o) = E[A^{\kappa-1}] = o^{\kappa+1} \Gamma(\kappa).$$

Lemma 2: ET of Caputo fractional (CF) derivative for

$$\kappa > 0, m-1 < \kappa \leq m, m \in N,$$

is

$$E[D_\tau^\kappa \Phi(v, \tau)] = o^{m-\kappa} \left[\frac{T(v, o)}{o^m} - \frac{\Phi(v, 0)}{o^{m-2}} - \frac{\frac{\partial \Phi(v, 0)}{\partial \tau}}{o^{m-3}} - \dots \right]$$

or

$$E[D_\tau^\kappa \Phi(v, \tau)] = \frac{1}{o^\kappa} E[\Phi(v, \tau)] - \sum_{k=0}^{m-1} \frac{\partial^k \Phi(v, 0)}{\partial \tau^k} o^{2-\kappa+k},$$

$$m-1 < \kappa \leq m,$$

The following is the definition of the Mittag-Leffler functions:

$$\epsilon_\kappa(\tau) = \sum_{n=0}^\infty \frac{\tau^n}{\Gamma(n\kappa+1)}, \epsilon_{\kappa, \lambda}(\tau) = \sum_{n=0}^\infty \frac{\tau^n}{\Gamma(n\kappa+\lambda)}.$$

Where Γ is the gamma function and κ, λ are complex parameters with real part are zero.

Lemma 3:

If $\kappa, \lambda > 0, a \in \mathbb{C}$ and $\frac{1}{o^\kappa} > |a|$, then:

$$E^{-1}\left[\frac{o^{\lambda+1}}{1+ao^\kappa}\right] = \tau^{\lambda-1} \epsilon_{\kappa, \lambda}(-a\tau^\kappa).$$

Proof:

$$\frac{o^{\lambda+1}}{1+ao^\kappa} = o^{\lambda+1} \frac{1}{1+ao^\kappa} = o^{\lambda+1} \sum_{n=0}^\infty (-a)^n (o^\kappa)^n$$

$$= \sum_{n=0}^\infty (-a)^n o^{n\kappa+\lambda+1}$$

Then

$$E^{-1} \left[\frac{o^{\lambda+1}}{1+o^\kappa} \right] = E^{-1} \left[\sum_{n=0}^{\infty} (-a)^n o^{n\kappa+\lambda+1} \right] =$$

$$\sum_{n=0}^{\infty} \frac{(-a)^n \tau^{n\kappa+\lambda-1}}{\Gamma(n\kappa+\lambda)} =$$

$$\tau^{\lambda-1} \sum_{n=0}^{\infty} \frac{(-a\tau^\kappa)^n}{\Gamma(n\kappa+\lambda)} = \tau^{\lambda-1} \in_{\kappa,\lambda} (-a\tau^\kappa).$$

3. THE NOVEL ANALYTICAL METHOED

The primary concept of the suggested plan of study will be made clearer by applying the SFPDE in the following ways,

$$D_\tau^\kappa P(v, \tau) + R_1[P, Q, P_v, Q_v] + N_1[P, Q, P_v, Q_v] = g_1(v, \tau),$$

$$D_\tau^\kappa Q(v, \tau) + R_2[P, Q, P_v, Q_v] + N_2[P, Q, P_v, Q_v] = g_2(v, \tau),$$

$$0 < \kappa \leq 1, 0 \leq v \leq k, \tau > 0, k > 0 \quad (2)$$

with the ICs,

$$P(v, 0) = h_1(v), Q(v, 0) = h_2(v) \quad (3)$$

Where D_τ^κ is a Caputo FD, R_1, R_2, N_1, N_2 are linear and nonlinear operators, and $g_1(v, \tau), g_2(v, \tau)$ are inhomogeneous terms.

Taking ET of Eqs. (2), to get:

$$E[D_\tau^\kappa P(v, \tau)] + E\{R_1[P, Q, P_v, Q_v] + N_1[P, Q, P_v, Q_v]\} = E[g_1(v, \tau)],$$

$$E[D_\tau^\kappa Q(v, \tau)] + E\{R_2[P, Q, P_v, Q_v] + N_2[P, Q, P_v, Q_v]\} = E[g_2(v, \tau)], \quad (4)$$

$$E(P(v, \tau)) = o^2 h_1(x) - o^\kappa E\{R_1[P, Q, P_v, Q_v] + N_1[P, Q, P_v, Q_v] - g_1(v, \tau)\},$$

$$E(Q(v, \tau)) = o^2 h_2(x) - o^\kappa E\{R_2[P, Q, P_v, Q_v] + N_2[P, Q, P_v, Q_v] - g_2(v, \tau)\}.$$

Solutions of Eqs. (2), should be performed using the series form that follows:

$$P(v, \tau) = \sum_{n=0}^{\infty} P_n(v, \tau), Q(v, \tau) = \sum_{n=0}^{\infty} Q_n(v, \tau). \quad (5)$$

Using Eqs. (5) and the inverse of ET to Eqs. (4), to get:

$$\sum_{n=0}^{\infty} P(v, \tau) = G_1(v, \tau) - E^{-1} \{o^\kappa E[R_1[P, Q, P_v, Q_v] + N_1[P, Q, P_v, Q_v]]\},$$

$$\sum_{n=0}^{\infty} Q(v, \tau) = G_2(v, \tau) - E^{-1} \{o^\kappa E[R_2[P, Q, P_v, Q_v] + N_2[P, Q, P_v, Q_v]]\} \quad (6)$$

$G_1(v, \tau), G_2(v, \tau)$ are the terms found that come from the source terms and the ICs.

This approach is dependent on how we select the initial iterations P_0, Q_0 that yield the precise solutions in a constrained amount of steps. The solutions $P(v, \tau), Q(v, \tau)$, can be found iteratively using the following relations

$$P_{n+1}(v, \tau) = E^{-1} \left\{ o^\kappa E \left[R_1[P_n, Q_n, (P_n)_v, (Q_n)_v] + N_1[P_n, Q_n, (P_n)_v, (Q_n)_v] \right] \right\}$$

$$P_0(v, \tau) = G_1(v, \tau),$$

$$Q_{n+1}(v, \tau) = E^{-1} \left\{ o^\kappa E \left[R_2[P_n, Q_n, (P_n)_v, (Q_n)_v] + N_2[P_n, Q_n, (P_n)_v, (Q_n)_v] \right] \right\} \quad (7)$$

From Eqs. (7) and (4), we can determine that:

$$P_0, P_1, P_2, \dots, Q_0, Q_1, Q_2, \dots$$

using Eqs. (5) to find the solution.

4. ILLUSTRATIVE EXAMPLES

Using the process described in this study, we produced the exact solutions for FCSNLPDEs and compared them with the known exact solutions.

Example 1: Consider the FCSNLPDE

$$D_\tau^\kappa P(v, \tau) + P - 2PP_v + (PQ)_v = 0, 0 < \kappa \leq 1,$$

$$D_\tau^\kappa Q(v, \tau) + Q - 2QQ_v + (PQ)_v = 0 \quad (8)$$

with the ICs

$$P(v, 0) = \sin v, Q(v, 0) = \sin v \quad (9)$$

Using the ET of Eqs. (8) and ICs, to get:

$$\frac{1}{o^\kappa} E[P] - P(v, 0)o^{2-\kappa} + E[P] = E[2PP_v - (PQ)_v],$$

$$\frac{1}{o^\kappa} E[Q] - Q(v, 0)o^{2-\kappa} + E[Q] = E[2QQ_v - (PQ)_v],$$

$$E[P] = \frac{o^2}{1+o^\kappa} \sin v + \frac{o^\kappa}{1+o^\kappa} E[2PP_v - (PQ)_v],$$

$$E[Q] = \frac{o^2}{1+o^\kappa} \sin v + \frac{o^\kappa}{1+o^\kappa} E[2QQ_v - (PQ)_v]$$

Inverse ET shows:

$$P(v, \tau) = \in_{\kappa} (-\tau^\kappa) \sin v + E^{-1} \left\{ \frac{o^\kappa}{1+o^\kappa} E[2PP_v - (PQ)_v] \right\},$$

$$Q(v, \tau) = \in_{\kappa} (-\tau^\kappa) \sin v + E^{-1} \left\{ \frac{o^\kappa}{1+o^\kappa} E[2QQ_v - (PQ)_v] \right\}$$

The iteration formulas that utilize an initial approximation are as follows:

$$P_{n+1}(v, \tau) = E^{-1} \left\{ \frac{o^\kappa}{1+o^\kappa} E[2P_n(P_n)_v - (P_n Q_n)_v] \right\},$$

$$Q_{n+1}(v, \tau) = E^{-1} \left\{ \frac{o^\kappa}{1+o^\kappa} E[2Q_n(Q_n)_v - (P_n Q_n)_v] \right\},$$

$$P_0(v, \tau) = \in_{\kappa} (-\tau^\kappa) \sin v, Q_0(v, \tau) = \in_{\kappa} (-\tau^\kappa) \sin v, \quad (10)$$

Then we find:

$$P_1(v, \tau) = E^{-1} \left\{ \frac{o^\kappa}{1+o^\kappa} E[0] \right\} = 0, Q_1(v, \tau) = E^{-1} \left\{ \frac{o^\kappa}{1+o^\kappa} E[0] \right\} = 0,$$

Then, using (5), to get:

$$P(v, \tau) = \in_{\kappa} (-\tau^\kappa) \sin v, Q(v, \tau) = \in_{\kappa} (-\tau^\kappa) \sin v,$$

If $\kappa = 1$, then:

$$P(v, \tau) = e^{-\tau} \sin v, Q(v, \tau) = e^{-\tau} \sin v$$

Example 2: Consider the FCSNLPDE

$$D_\tau^\kappa P(v, \zeta, \tau) + Q_v K_\zeta - Q_\zeta K_v = -P, 0 < \kappa \leq 1,$$

$$D_\tau^\kappa Q(v, \zeta, \tau) + P_v K_\zeta + P_\zeta K_v = Q,$$

$$D_\tau^\kappa K(v, \zeta, \tau) + P_v Q_\zeta + P_\zeta Q_v = K \quad (11)$$

with the IC and BCs

$$P(v, \zeta, 0) = e^{v+\zeta}, Q(v, \zeta, 0) = e^{v-\zeta}, K(v, \zeta, 0) = e^{\zeta-v} \quad (12)$$

Using similar steps as in example 1, to obtain:

$$P(v, \zeta, \tau) = \in_{\kappa} (-\tau^\kappa) e^{v+\zeta} + E^{-1} \left\{ \frac{o^\kappa}{1+o^\kappa} E[Q_v K_\zeta - Q_\zeta K_v] \right\},$$

$$Q(v, \zeta, \tau) = \epsilon_{\kappa} (\tau^{\kappa}) e^{v-\zeta} - E^{-1} \left\{ \frac{o^{\kappa}}{1-o^{\kappa}} E[P_v K_{\zeta} + P_{\zeta} K_v] \right\},$$

$$K(v, \zeta, \tau) = \epsilon_{\kappa} (\tau^{\kappa}) e^{\zeta-v} - E^{-1} \left\{ \frac{o^{\kappa}}{1-o^{\kappa}} E[P_v Q_{\zeta} + P_{\zeta} Q_v] \right\}$$

The recurring connections are as follows:

$$P_{n+1}(v, \zeta, \tau) = E^{-1} \left\{ \frac{o^{\kappa}}{1+o^{\kappa}} E[(Q_n)_v (K_n)_{\zeta} - (Q_n)_{\zeta} (K_n)_v] \right\} \\ \Rightarrow P_0 = \epsilon_{\kappa} (-\tau^{\kappa}) e^{v+\zeta}$$

$$Q_{n+1}(v, \zeta, \tau) = -E^{-1} \left\{ \frac{o^{\kappa}}{1-o^{\kappa}} E[(P_n)_v (K_n)_{\zeta} + (P_n)_{\zeta} (K_n)_v] \right\} \\ \Rightarrow Q_0 = \epsilon_{\kappa} (\tau^{\kappa}) e^{v-\zeta},$$

$$K_{n+1}(v, \zeta, \tau) = -E^{-1} \left\{ \frac{o^{\kappa}}{1-o^{\kappa}} E[(Q_n)_{\zeta} (P_n)_v + (Q_n)_v (P_n)_{\zeta}] \right\} \\ \Rightarrow K_0 = \epsilon_{\kappa} (\tau^{\kappa}) e^{\zeta-v} \quad (13)$$

From Eqs. (13), we get:

$$P_1 = E^{-1} \left\{ \frac{o^{\kappa}}{1+o^{\kappa}} E[0] \right\} = 0, Q_1 = -E^{-1} \left\{ \frac{o^{\kappa}}{1-o^{\kappa}} E[0] \right\} = 0,$$

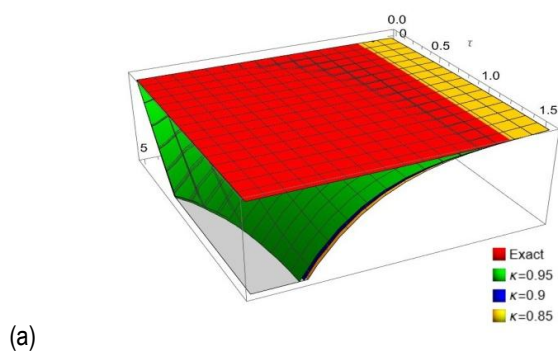
$$K_1 = -E^{-1} \left\{ \frac{o^{\kappa}}{1-o^{\kappa}} E[0] \right\} = 0$$

Using Eq. (11) to get:

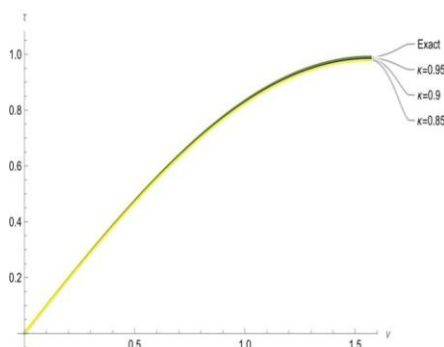
$$P = \epsilon_{\kappa} (-\tau^{\kappa}) e^{v+\zeta}, Q = \epsilon_{\kappa} (\tau^{\kappa}) e^{v-\zeta}, K = \epsilon_{\kappa} (\tau^{\kappa}) e^{\zeta-v}$$

The subsequent exact solutions can be obtained if $\kappa = 1$,

$$P(v, \zeta, \tau) = e^{v+\zeta-\tau}, Q(v, \zeta, \tau) = e^{v-\zeta+\tau}, K(v, \zeta, \tau) = e^{\zeta-v+\tau}.$$

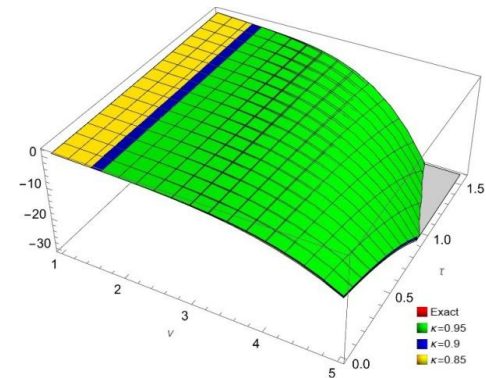


(a)

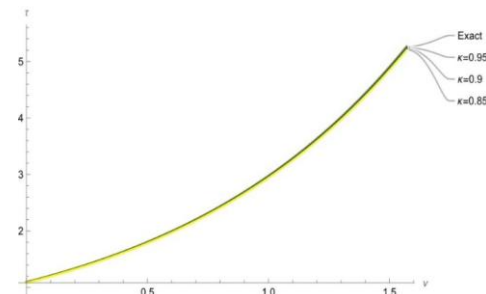


(b)

Fig.1. (a) Comparing the provided method's 3D representations of the solutions to the exact solutions for Eq. (8), the outcome in comparison to the exact solutions, (b) represent the solutions at $\kappa = 0.95, 0.9, 0.85$

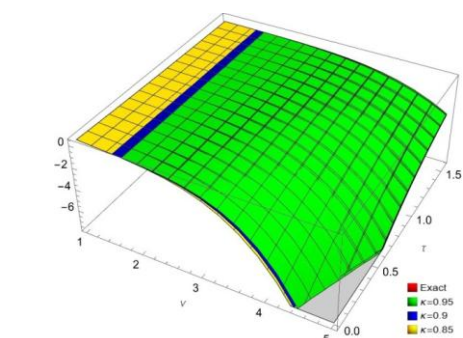


a)

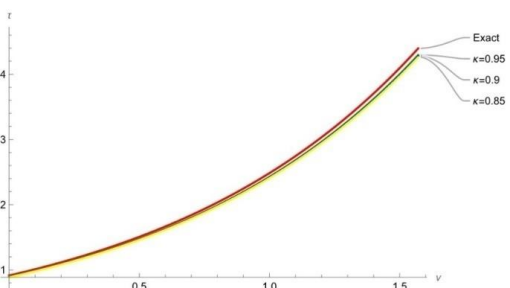


(b)

Fig. 2. (a) Comparing the provided method's 3D representations of the solutions to the exact solutions for Eq. (11), $P(v, \zeta, \tau)$ the outcome in comparison to the exact solutions, (b) represent the solutions at $\kappa = 0.95, 0.9, 0.85$



(a)



(b)

Fig. 3. (a) Comparing the provided method's 3D representations of the solutions to the exact solutions for Eq. (11), $Q(v, \zeta, \tau)$ the outcome in comparison to the exact solutions, (b) represent the solutions at $\kappa = 0.95, 0.9, 0.85$

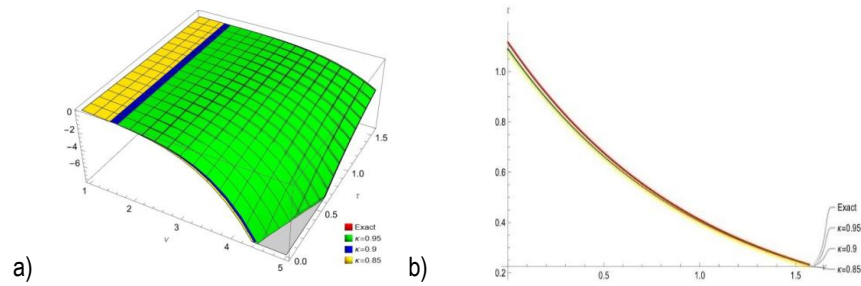


Fig.4. (a) Comparing the provided method's 3D representations of the solutions to the exact solutions for Eq. (11), $K(v, \zeta, \tau)$ the outcome in comparison to the exact solutions, (b) represent the solutions at $\kappa = 0.95, 0.9, 0.85$

Tab. 1. The comparison of the exact and approximate solutions yields the numerical result for example (1)

	τ	v	$\kappa = 0.85$	$\kappa = 0.9$	$\kappa = 0.95$	<i>Exact</i>	<i>Error</i>
		0	0.	0.	0.	0.	0.
		$\frac{\pi}{6}$	0.489578	0.491835	0.493619	0.495025	0.00140562
$P(v, \tau)$ $Q(v, \tau)$	0.01	$\frac{\pi}{3}$	0.847973	0.851883	0.854974	0.857408	0.00243461
		$\frac{\pi}{2}$	0.979155	0.98367	0.987239	0.99005	0.00281124

Tab. 2. The comparison of the exact and approximate solutions yields the numerical result for example (2)

	ζ	τ	v	$\kappa = 0.85$	$\kappa = 0.9$	$\kappa = 0.95$	<i>Exact</i>	<i>Error</i>
			0	1.08213	1.08712	1.09107	1.09417	0.0031069
			$\frac{\pi}{6}$	1.82674	1.83516	1.84182	1.84707	0.00524474
$P(v, \zeta, \tau)$	0.1	0.01	$\frac{\pi}{3}$	3.08371	3.09793	3.10916	3.11802	0.0088536
			$\frac{\pi}{2}$	5.20558	5.22958	5.24855	5.2635	0.0149457

Tab. 3. The comparison of the exact and approximate solutions yields the numerical result for example (2)

	ζ	τ	v	$\kappa = 0.85$	$\kappa = 0.9$	$\kappa = 0.95$	<i>Exact</i>	<i>Error</i>
			0	0.885976	0.890061	0.89329	0.913931	0.0206408
			$\frac{\pi}{6}$	1.49561	1.50251	1.50796	1.5428	0.0348435
$Q(v, \zeta, \tau)$	0.1	0.01	$\frac{\pi}{3}$	2.52473	2.53637	2.54557	2.60439	0.058819
			$\frac{\pi}{2}$	4.26197	4.28162	4.29715	4.39645	0.0992919

Tab. 4. The comparison of the exact and approximate solutions yields the numerical result for example (2)

	ζ	τ	v	$\kappa = 0.85$	$\kappa = 0.9$	$\kappa = 0.95$	<i>Exact</i>	<i>Error</i>
			0	1.08213	1.08712	1.09107	1.11628	0.0252107
			$\frac{\pi}{6}$	0.64104	0.643995	0.646332	0.661266	0.0149344
$K(v, \zeta, \tau)$	0.1	0.01	$\frac{\pi}{3}$	0.379742	0.381493	0.382877	0.391724	0.00884693
			$\frac{\pi}{2}$	0.224954	0.225991	0.226811	0.232051	0.00524079

4. NUMERICAL SIMULATIONS

Using a new analytical technique based on ET, this study solves the FCSNLPDEs with ICs; the outcomes of previous approaches are not comparable to those of the present techniques. Multiple parameter values are provided by Equations (8) and (11) in order to give a range of solutions. By allowing the arbitrary parameters to have varying values in the solutions, a range of solutions may be generated. The collected replies are grouped into categories. Additionally, 2D and 3D visual representations are created. These plots may be described using the following details: Figures 1, 2, 3, and 4 show different arrangements of lone waves. Figure 1 was produced for the values in Eq. (8), showing: $at\kappa = 0.95, 0.9, 0.85, \tau = 0.01$. This mixture falls within the periodic category, in Eq. (11). Figures 2, 3 and 4 are created using $\kappa = 0.95, 0.9, 0.85, \tau = 0.01$.

The exact solution, which was determined by comparing the values of the exact and approximate solutions of FCSNLPDEs discovered in this problem for various values of the variables, $0 \leq v \leq \frac{\pi}{2}, \tau = 0.01$, is provided in Table 1. Tables 2, 3 and 4 are contains the approximate solutions, which was ascertained by

comparing the values of the exact solutions of FCSNLPDEs for various values of the variables, $0 < v \leq \frac{\pi}{2}, \zeta = 0.1, \tau = 0.01$. It was discovered that the proposed methods were effective and feasible. The simulations were conducted and the results analysed using the Wolfram Mathematica software.

5. CONCLUSION

In order to solve FCSNLPDEs using ICs, we created the ET in this study. To demonstrate the usefulness of the proposed method, two examples are provided. It may be possible to handle the solutions in a very basic manner.

The method may be used to tackle a range of initial value problems because to its exceptional ability to solve FCSNLPDEs for various values of fractional orders. Furthermore, 2D and 3D graphs and tables were used to show how the recommended strategy influenced the results. This clearly shows that the new method considerably outperforms the fields when compared to the older methods. In this article we just want to demonstrate the effectiveness of the method discussed and in the near future we will compare this method with other methods to confirm its effectiveness in solving other fractional equations. Also we will consider in future studies the possibility of adapting this approach to other complex systems or fuzzy differential equations. There will be further study done on how to apply this approach to more complex problems, how to integrate it with more sophisticated computer techniques, and how to apply it in the actual world for problems like biological system simulation and epidemic prediction. This advancement creates new avenues for identifying novel solutions to challenging problems in science. Future research will apply the proposed approach to biological simulations and epidemic prediction, extend it to more intricate and stochastic systems, and increase computer efficiency using advanced techniques.

REFERENCES

- Podlubny I. Fractional differential equations. Academic Press. San Diego; 1999.
- Podlubny I. Geometric and physical interpretation of fractional

- integration and fractional differentiation. *Fract. Calculus. Appl. Anal.* 2002;5:367-386.
- He J. Nonlinear oscillation with fractional derivative and its applications, International Conference on Vibrating Engineering. Dalian. China. 1998;288-291.
- He J. Some applications of nonlinear fractional differential equations and their approximations. *Bull. Sci. Technol.* 1999; 15:86-90.
- He J. Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Comput. Meth-ods. Appl. Mech. Eng.* 1998;167:57-68.
- Grigorenko I, Grigorenko E. Chaotic dynamics of the fractional lorenz system. *Phys. Rev. Lett.* 2003; 91:034101.
- Mainardi F. Fractional calculus: some basic problems in continuum and statistical mechanics. *Fractals and fractional calculus in continuum mechanics.* Springer-Verlag, New York. 1997;291-348.
- Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Elsevier, San Diego; 2006.
- Momani S, Shawagfeh NT. Decomposition method for solving fractional Riccati differential equations. *Appl. Math. Comput.* 2006; 182:1083-1092.
- Momani S, Noor MA. Numerical methods for fourth-order fractional integrodifferential equations, *Appl. Math. Comput.* 2006;182:754-760.
- Gejji VD, Jafari H. Solving a multi-order fractional differential equation. *Appl. Math. Comput.* 2007;189:541-548.
- Ray SS, Chaudhuri KS, Bera RK. Analytical approximate solution of nonlinear dynamic system containing fractional derivative by modified decomposition method. *Appl. Math. Comput.* 2006;182:544-552.
- Wang Q. Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method. *Appl. Math. Comput.* 2006; 182:1048-1055.
- Iskakov K, Alam MM, Ahmad A, Saifullah S, Akgül A, Yılmaz G. Dynamical study of a novel 4D hyperchaotic system: An integer and fractional order analysis. *Mathematics and Computers in Simulation.* 2023;208: 219-245.
- Alqahtani RT, Ahmad S, Akgül A. Dynamical Analysis of Bio-Ethanol Production Model under Generalized Nonlocal Operator in Caputo Sense. *Mathematics.* 2021; 9(19):2370. <https://doi.org/10.3390/math9192370>
- Ahmad S, Ullah A, Akgül A, Jarad F. A hybrid analytical technique for solving nonlinear fractional order PDEs of power law kernel: Application to KdV and Fornberg-Witham equations [J]. *AIMS Mathematics.* 2022; 7(5): 9389-9404. <https://doi.org/10.3934/math.2022521>
- Alqahtani RT, Ahmad S, Akgül A. Mathematical Analysis of Biodegradation Model under Nonlocal Operator in Caputo Sense. *Mathematics.* 2021; 9(21):2787. <https://doi.org/10.3390/math9212787>
- Ahmad S, Saifullah S. Analysis of the seventh-order Caputo fractional KdV equation: applications to the Sawada-Kotera-Ito and Lax equations, *Commun. Theor. Phys.* 2023; 75(8): 085002. <https://doi.org/10.1088/1572-9494/acded7>
- Hilal EMA, Elzaki TM. Solution of Nonlinear Partial Differential Equations by New Laplace Variational Iteration Method. *Journal of Function Spaces.* 2014;1-5. <http://dx.doi.org/10.1155/2014/790714>
- Elzaki TM, Biazar J. Homotopy Perturbation Method and Elzaki Transform for Solving System of Nonlinear Partial Differential Equations. *World Applied Sciences Journal.* 2013. <https://doi.org/10.5829/idosi.wasj.2013.24.07.1041>
- Elzaki TM. Application of Projected Differential Transform Method on Nonlinear Partial Differential Equations with Proportional Delay in One Variable. *World Applied Sciences Journal.* 2014. <https://doi.org/10.5829/idosi.wasj.2014.30.03.1841>
- Masood S, Khan HH, Shah R, Mustafa S, Khan Q, Arif M, Tchier F, Singh G. A New Modified Technique of Adomian Decomposition Method for Fractional Diffusion Equations with Initial-Boundary Conditions. *Journal of Function Spaces.* 2022; 6890517. <https://doi.org/10.1155/2022/6890517>
- Ige OE, Oderinu RA, Elzaki TM. Numerical Simulation Of The Nonlinear Coupled Jaulent-Miodek Equation By Elzaki Transform-Adomian Polynomial Method. *Advances in Mathematics: Scientific Journal.* 2020;9(12): 10335-10355. <https://doi.org/10.37418/amsj.9.12.25>
- Ige OE, Oderinu RA, Elzaki TM. Adomian Polynomial And Elzaki

- Transform Method For Solving Klein Gordon Equations. International Journal of Applied Mathematics. 2019; 32(3): 451-468.
<http://dx.doi.org/10.12732/ijam.v32i3.7>
25. Alderremy AA, Elzaki TM, Chamekh M. Modified Adomian Decomposition Method to Solve Generalized Emden–Fowler Systems for Singular IVP. Hindawi. Mathematical Problems in Engineering. 2019; 6097095. <https://doi.org/10.1155/2019/6097095>
 26. Elzaki TM, Ishag AA. Modified Laplace Transform and Ordinary Differential Equations with Variable Coefficients. World Engineering & Applied Sciences Journal. 2019; 10 (3): 79-84.
<https://doi.org/10.5829/idosi.weasj.2019.79.84>
 27. Elzaki TM, Chamekh M, Ahmed SA. Modified Integral Transform for Solving Benney-Luke and Singular Pseudo-Hyperbolic Equations. Acta Mechanica et Automatica. 2024;18(1).
[https://doi.org/10.2478/ama-\(2024\)-0018](https://doi.org/10.2478/ama-(2024)-0018)
 28. Ahmed SA, Saade R, Qazza A, Elzaki TM. Applying Conformable Double Sumudu–Elzaki Approach to Solve Nonlinear Fractional Problems. Progr. Fract. Differ. Appl. 2024;10(2): 271-286.
<http://dx.doi.org/10.18576/pfda/100208>
 29. Elzaki TM, Mohamed MZ. A Novel Analytical Method For The Exact Solution Of The Fractional-Order Biological Population Model. Acta Mechanica et Automatica. 2024; 18(3):564-570.
<https://doi.org/10.2478/ama-2024-0059>
 30. Chamekh M, Latrach MA, Elzaki TM. Novel Integral Transform Treating Some Ψ -Fractional Derivative Equations. Acta Mechanica et Automatica. 2024; 18 (3). <https://doi.org/10.2478/ama-2024-0060>
 31. Elzaki TM, Ahmed SA. Novel approach for solving fractional partial differential equations using conformable Elzaki Transform. J. Umm Al-Qura Univ. Appl. Sci. 2024. <https://doi.org/10.1007/s43994-024-00188-0>
 32. Magzoub M, Elzaki TM, Chamekh M. An innovative method for solving linear and nonlinear fractional telegraph equations. Advances in Differential Equations and Control Processes. 2024; 31(4): 651-671. <https://doi.org/10.17654/0974324324033>

Mohammed E. A. Rabie:  <https://orcid.org/0009-0002-0446-8033>

Tarig M. Elzaki:  <https://orcid.org/0000-0002-6946-9267>



This work is licensed under the Creative Commons BY-NC-ND 4.0 license.